A New Family of Gamma- Exponential-Exponential Distribution Using T-X Approach

Olayinka K. P.[1], Akinyele T. W.[2] and *Ajayi E. D.[2]

[1]Department of Statistics, Federal Polytechnic, Ado-Ekiti

[2]Department of Statistics, Ekiti State University, Ado-Ekiti

***Corresponding Author**

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ABSTRACT

A new family of continuous distributions called the Gamma-Exponential- Exponential distribution was developed by modifying a newly generated continuous T-X family of distribution called the Gamma-Exponential -X distribution. Statistical properties such as moment generation function, characteristic function, skewness and kurtosis of the newly developed distribution was derived in this paper. Maximum likelihood estimation approach was used to estimate the parameter of the distributions. The distribution is suitable to fit both right and left skewed data. Goodness of fit statistics of the new distribution were computed using an acute leukemia data and compared with that of existing estimators. Gamma-Exponential- Exponential distribution performed better than these well-established estimators. This newly developed Gamma-Exponential-Exponential distribution has proven its efficiency as a real-world analytical model by exhibiting adaptability and efficacy in model fits.

Keywords: Distribution, Gamma, Exponential, moment generating function, maximum likelihood

INTRODUCTION

Statistical distributions are very useful in describing real-world phenomenon. Although many distributions have been developed but there is always a room for new distributions which are either more flexible in term of fitting a specific real-world scenario. This attempt has motivated researchers to seek and develop new flexible distributions. As a result, many new distributions have been developed and studied. From the past years, there is a growing trend of generating new families of distributions from existing distribution by adding one or more additional parameter(s) to the baseline distribution to study the behavior of the shapes of density and hazard rate, and for checking the goodness-of-fit, we intend to introduce Gamma-Exponential-Exponential distribution (GEED), a novel member of the Gamma-Exponential -X family of generalized distributions. Several researchers have generated new adaptable distributions from existing distributions using various modification techniques to increase their flexibility in modeling data.

These adaptable distributions are created by adding extra parameters to the baseline distribution with generators or combining two distributions (Ali, *et al*., 2021). These modified distributions can model data sets efficiently and, in most cases, provide the best fit to data sets when applied because they have more parameters and are more adaptable than their baseline distributions.

Introducing an extra parameter to an existing distribution has proven beneficial in studying tail properties and improving the goodness of fit of the compound distribution. Recent emerging data of interest exhibit non-normal characteristics such as high or moderate skewness and high kurtosis, which existing distributions cannot accurately model. Cancer continues to impact the world significantly, and there was recently a Covid-19 virus that spread across the globe. The data sets representing these diseases mortality or infection rate

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exhibit some of the stated features. The results of fitting these data sets with traditional distributions may be unreliable (Albalawi, *et al*., 2022).

Many applications, such as lifetime analysis require extended forms of these distributions. Most statistical distribution modeling approaches are concerned with determining which probability distribution best represents the data. No single probability distribution, however, can fit all types of data. As a result, new classical distributions must be developed or created (Nasiru, 2018).

The gamma distribution is a family of two-parameter continuous probability distributions. The gamma distribution is a category of continuous probability distribution that is skewed to the right. Insurance claim sizes have been modelled using the gamma distribution. (Philip, 2007) and rainfalls (Aksoy, 2000), which means that aggregate insurance claims and rainfall accumulation in a reservoir are modelled by a gamma process, similar to how the exponential distribution generates a Poisson process.

The exponential distribution is a continuous statistical distribution that estimates the times when events occur. These occurrences are independent and occur at a constant average rate. In other words, it is used to predict the waiting time of a person before an event occurs. It is a continuous counterpart of geometric distribution. It is a memoryless random distribution with many small and few large values. It differs from the Poisson distribution in that it predicts the number of times an event occurs in a given period rather than the time interval.

METHODOLOGY

Gamma-Exponential distribution

Let X be continuous independent random variables such that; $x \sim \text{GED}(x,,\sigma,\beta,\lambda)$ and,

let $f(x)$ be the probability density function of Gamma-Exponential distribution given as;

$$F(x) = \frac{\lambda^{\alpha} X^{\alpha-1} e^{-\frac{\lambda x}{\beta}}}{\Gamma \propto \beta^{\alpha}} \text{, where } x > 0,\ \alpha, \beta, \lambda \geq 0 \qquad 2.10$$

$$g(x) = \frac{f(x)}{1-F(x)}\, r(x) \quad \text{where } t = -\log(1-F(x)) \qquad 2.11$$

$$r(t) = \frac{\lambda^{\alpha} t^{\alpha-1} e^{-\frac{\lambda t}{\beta}}}{\Gamma \alpha \beta^{\alpha}} \qquad 2.12$$

$$r(x) = \frac{\lambda^{\alpha}[-\log(1-F(x))]^{\alpha-1}(1-F(x))^{\lambda/\alpha}}{\Gamma \alpha \beta^{\alpha}} \qquad 2.13$$

$$g(x) = \frac{f(x)}{1-f(x)} \lambda^{\alpha}[-\log(1-F(x))]^{\alpha-1} (1 - F(x))^{\lambda/\beta} \frac{1}{\sqrt{\alpha}\beta^{\alpha}} \qquad 2.14$$

$$g(x) = \frac{\lambda e^{-\lambda x}}{\Gamma \alpha \beta^{\alpha}} \lambda^{\alpha}[-\log(1-F(x))]^{\alpha-1} (1\text{-F(x)}^{\lambda/\beta-1}) \qquad 2.15$$

$$F(x) = \int_0^{\infty} f(x)dx = \int_0^{\infty} \lambda e^{-\lambda x} = \lambda \int_0^{\infty} e^{-\lambda x} dx = 1 - e^{-\lambda x} \qquad 2.16$$

$$= \frac{\lambda e^{-\lambda x}}{\Gamma \alpha \beta^{\alpha}} \lambda^{\alpha}[-\log(1-F(x))]^{\alpha-1} (1\text{-F(x)})^{\lambda/\beta^{-1}} \qquad 2.17$$

$$= \frac{\lambda e^{-\lambda x}}{\Gamma \alpha \beta^{\alpha}} \lambda^{\alpha}[-\log e^{-\lambda x}]^{\alpha-1} (e^{-\lambda x})^{\lambda/\beta-1} \qquad 2.18$$

The probability density function of Gamma-Exponential-Exponential distribution is given as:

ISSN No. 2321-2705 | DOI: 10.51244/IJRSI |Volume XIII Issue VI June 2026

$f(x)_{GEED} = \frac{\lambda^{2\alpha}\, x^{\alpha-1} e^{\frac{-\lambda^2}{\beta}}}{\Gamma\alpha\beta^{\alpha}}$, where $x > 0, \alpha, \beta, \lambda \geq 0$ 2.19

Moment Generating Function of Gamma-Exponential –Exponential Distribution

$$M_X^{(t)} = E(e^{tx}) = \int_0^{\infty} e^{tx}\ f(x,\alpha,\beta,\lambda)dx \qquad 2.20$$

$$M_X^{(t)} = \int_0^{\infty} \frac{e^{tu}\, \lambda^{2\alpha}\, x^{\alpha-1}\, e^{\frac{-\lambda 2x}{\beta}}}{\Gamma\alpha\beta^{\alpha}} \qquad 2.21$$

$$M_X^{(t)} = \frac{1}{\Gamma\alpha\beta^{\alpha}} \int_0^{\infty} \lambda^{2\alpha}\, x^{\alpha-1} e^{x}(\frac{t-\lambda^2}{\beta})dx \qquad 2.22$$

Let $u = x(\frac{t-\lambda^2}{\beta})$, $x = \frac{u}{\frac{t-\lambda^2}{\beta}}$, $dx = \frac{du}{\frac{t-\lambda^2}{\beta}}$

$$M_X^{(t)} = \frac{\lambda^{2\alpha}}{\Gamma\alpha\beta^{\alpha}} \int_0^{\infty} \left(\frac{u}{\frac{t-\lambda^2}{\beta}}\right)^{\alpha-1} e^{u}\ \frac{du}{\frac{t-\lambda^2}{\beta}} \qquad 2.23$$

$$M_X^{(t)} = \frac{\lambda^{2\alpha}}{\Gamma\alpha\beta^{\alpha}} \int_0^{\infty} u^{\alpha-1} \left(\frac{1}{\frac{t-\lambda^2}{\beta}}\right)^{\alpha-1} \frac{1}{\frac{t-\lambda^2}{\beta}}\ e^{u}du \qquad 2.24$$

$$M_X^{(t)} = \frac{\lambda^{2\alpha}}{\Gamma\alpha\beta^{\alpha}} \int_0^{\infty} u^{\alpha-1} \left(\frac{1}{\frac{t-\lambda^2}{\beta}}\right)^{\alpha} e^{u}du \qquad 2.25$$

$$M_X^{(t)} = \frac{\lambda^{2\alpha}}{\Gamma\alpha\beta^{\alpha}(t\frac{-\lambda^2}{\beta})^{\alpha}} \qquad 2.26$$

The moment generating function of Gamma-Exponential –Exponential distribution is given as:

$$\mu_x^{(t)} = \frac{\lambda^{2\alpha}}{\beta^{\alpha}(t\frac{-\lambda^2}{\beta})^{\alpha}} \qquad 2.27$$

Characteristics Function Of Gamma-Exponential-Exponential Distribution

$$\phi_X^{(it)} = E(e^{itu}) = \int_0^{\infty} e^{itu} f(x,\alpha,\beta,\lambda)dx \qquad 2.30$$

$$\phi_X^{(it)} = \int_0^{\infty} \frac{e^{itu}\, \lambda^{2\alpha}\, x^{\alpha-1}\, e^{\frac{-\lambda^2 x}{\beta}}}{\Gamma\alpha\beta^{\alpha}} dx \qquad 2.31$$

$$\phi_X^{(it)} = \frac{\lambda^{2\alpha}}{\Gamma\alpha\beta^{\alpha}} \int_0^{\infty} x^{\alpha-1}\ e^{x} \left(\frac{it-\lambda^2}{\beta}\right) dx \qquad 2.32$$

Let $u = x\left(\frac{it-\lambda^2}{\beta}\right)$, $x = \frac{\frac{u}{t-\lambda^2}}{\beta}$, $dx = \frac{du}{it\frac{-\lambda^2}{\beta}}\ \phi_X^{(it)}$

$$\phi_X^{(it)} = \frac{\lambda^{2\alpha}}{\Gamma\alpha\beta^{\alpha}} \int_0^{\infty} \left(\frac{1}{\frac{it-\lambda^2}{\beta}}\right) u^{\alpha-1}e^{u}du \qquad 2.33$$

$$\phi_X^{(it)} = \frac{\lambda^{2\alpha}}{\Gamma\alpha\beta^{\alpha}\left(it-\frac{\lambda^2}{\beta}\right)^{\alpha}} \int_0^{\infty} u^{\alpha-1}e^{u}du \qquad 2.34$$

ISSN No. 2321-2705 | DOI: 10.51244/IJRSI |Volume XIII Issue VI June 2026

$$\phi_X^{(it)} = \frac{\lambda^{2\alpha}}{\Gamma\alpha\beta^{\alpha}\left(it-\frac{\lambda^2}{\beta}\right)^{\alpha}} \quad \Gamma\alpha \qquad 2.35$$

The Characteristics function Of Gamma-Exponential-Exponential distribution is given as:

$$\Phi_x^{(it)} = \frac{\lambda^{2\alpha}}{\beta^{\alpha}\left(it-\frac{\lambda^2}{\beta}\right)^{\alpha}} \qquad 2.36$$

Skewness of Gamma-Exponential-Exponential Distribution

$$S = \frac{E(x-\mathcal{M})^3}{\delta^3} = \frac{\mathcal{M}^3}{\delta^3} = \frac{\mu_3{}^i - 3\mu_1{}^1\mu_2{}^i + \mu_1{}^i}{\delta^3} \qquad 2.41$$

$$= \frac{\frac{\alpha(\alpha+1)(\alpha+2)\beta^3}{\lambda^6} - 3\left(\frac{\alpha\beta}{\lambda^2}\left(\alpha(\alpha+1)\frac{\beta^2}{\lambda^4}\right)+2\left(\frac{\alpha^3\beta^3}{\lambda^6}\right)\right)}{\left[\sqrt{\frac{\alpha\beta^2}{\lambda^6}}\right]^3} \qquad 2.42$$

$$= \frac{\alpha\beta^{3(\alpha+1)(\alpha+2)-3\alpha^2\beta^3}(\alpha+1)+2\alpha^3\beta^3}{\lambda^6} \; X \; \frac{\lambda^6}{\alpha^{\frac{3}{2}}\beta^2} \qquad 2.43$$

$$= \frac{\alpha^2+2\alpha+\alpha+2-3\alpha+2\alpha^2}{\sqrt{\alpha}} \qquad 2.44$$

Skewness of GEED Distribution is given as: $\frac{2}{\sqrt{\alpha}}$

Kurtosis of Gamma-Exponential-Exponential Distribution

$$K = \frac{E(x-\mu)^4}{\sigma^4} = \frac{\mu^4}{\sigma^4} = \frac{\mu_4{}^1 - 4\mu_1{}^1\mu_3{}^1 + 6\mu_1{}^2\mu_2{}^1 - 3\mu_1{}^4}{\sigma^4} \qquad 2.50$$

$$K = \frac{\frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)\beta^4}{\lambda^8} - 4\left[\frac{\alpha\beta}{\lambda^4}\left(\frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)\beta^4}{\lambda^6}\right)\right] + \frac{6\alpha^2\beta^2}{\lambda^4}\left(\frac{\alpha\beta^2}{\lambda^4}(\alpha+1)\right)\frac{-3\alpha^4\beta^4}{\lambda^8}}{\left[\sqrt{\frac{\alpha\beta^2}{\lambda^4}}\right]^4} \qquad 2.51$$

$$K = \frac{\alpha\beta^4(\alpha+1)(\alpha+2)(\alpha+3) - 4\alpha^2\beta^4(\alpha+1)(\alpha+2) + 6\alpha^3\beta^4(\alpha+1) - 3\alpha^4\beta^4}{\lambda^8} \; X \; \frac{\lambda^8}{\alpha^2\beta^4} \qquad 2.52$$

$$K = \frac{\alpha^3+6\alpha^2+11\alpha+6-4\alpha^3-12\alpha^2-8\alpha+6\alpha^3+6\alpha^2-3\alpha^2}{\alpha} \qquad 2.53$$

Kurtosis of GEED Distribution is given as:

$$K = \frac{3\alpha+6}{\alpha} \qquad 2.54$$

Maximum Likelihood Estimation of Gamma-Exponential-Exponential Distribution

$$MLE = \left(\frac{\lambda^{2\alpha}}{\sqrt{\alpha\beta^{\alpha}}}\right)^n \prod_{i=1}^{n} x^{\alpha-1} \exp\left(\frac{-\lambda^2}{\beta} \sum_{i=1}^{n} x\underset{\sim}{i}\right) \qquad 2.60$$

Taking the natural log of both sides

$$\text{Log I} = 2\alpha n \log\lambda - n\log\sqrt{\alpha\beta^{\alpha}} + (\alpha-1)\sum_{i=1}^{n} xi \frac{-\lambda^2}{\beta}\sum_{i=1}^{n} x\underset{\sim}{i} \qquad 2.61$$

Therefore, the MLE satisfy the following normal equations;

ISSN No. 2321-2705 | DOI: 10.51244/IJRSI |Volume XIII Issue VI June 2026

$$\frac{dlog\ l}{d\lambda} = 2n\log\lambda - n\sqrt{\alpha^1\beta^2} + \sum_{i=1}^{n} x\underset{\sim}{i} \quad 2.62$$

$$\frac{dlog\ l}{d\lambda} = \frac{2\alpha n}{\lambda} - \frac{2\lambda}{\beta}\sum_{i=1}^{n} x\underset{\sim}{i} \quad 2.63$$

Empirical Comparison

The newly developed Gamma-Exponential-Exponential Distribution was applied to the data set used by Feigl and Zelen (1965). The data represent survival time, in weeks, of 33 patients suffering from Acute Myelogenous Leukaemia. The data are: 65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 8, 4, 3, 30, 4, 43.

Goodness-of-fit statistics: Akaike information criterion (AIC), Anderson-Darling (A^*) and Cramer-von Mises (W^*) of GEED distribution were compared with that of Gamma-Exponential (Ajewole *et al.*, 2013) distribution, Exponential distribution and Gamma distribution using the data above. Statistical analysis was done with Python software.

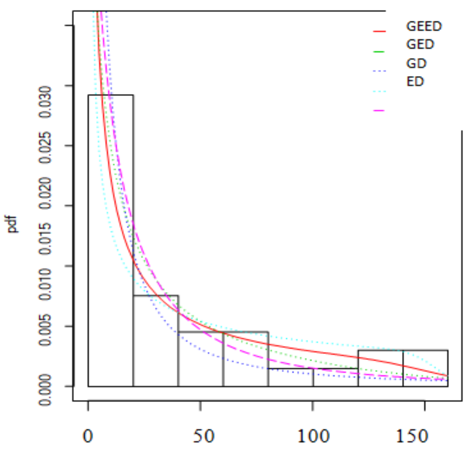
Table 1: Goodness of fit statistics

Distribution	AIC	A^*	W^*
GEED	309.63	0.4633	0.0666
GED	317.12	0.5657	0.0832
GD	319.70	0.7125	0.1128
ED	316.37	0.8984	0.1512

*GEED=Gamma-Exponential-Exponential distribution, *GED= Gamma-Exponential distribution, *GD= Gamma distribution, *ED= Exponential distribution

Table 1 reveals that GEED gives minimum values of the statistics AIC, A^* and W^* as compared to the three well-established distributions. Therefore, the proposed model GEED is better in performance than other competitive models.

Plots of fitted pdf. and cdf. of the models are shown in figures 1 and 2.



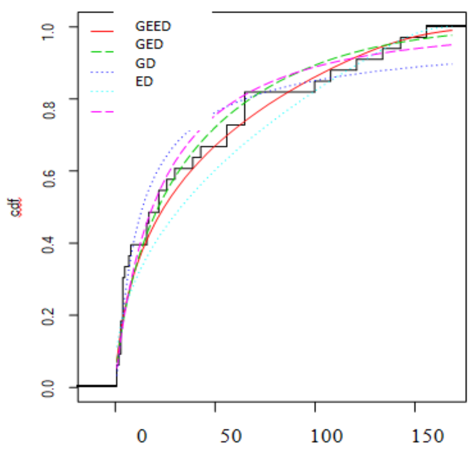


Fig 1: estimated pdf plot of the models, Fig 2: estimated cdf plot of the models

Figures 1 and 2 above confirms that the new model performs better than other competitive models.

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CONCLUSION

A new extension of the T-X family of the Gamma-Exponential-X distribution named Gamma-Exponential-Exponential distribution comprising three parameters has been developed and studied. The statistical properties of the new family of Gamma-Exponential-Exponential distribution has been derived, making it a proper distribution. The distribution showed that it is suitable to fit both right and left skewed data. Real life data set was analyzed to assess the performance of GEED utilizing goodness-of-fit measures such as the Akaike information criterion (AIC), Anderson-Darling (A^*) and Cramer-von Mises (W^*). According to the results, the Gamma-Exponential-Exponential distributions have the least AIC, A^* and W^* values. The Gamma-Exponential-Exponential distribution outperformed other existing distributions. The newly developed Gamma-Exponential-Exponential distribution has proven to be beneficial as a real-world analytical model by exhibiting adaptability and efficacy in model fits.

Conflict of Interests

The authors have not declared any conflict of interest.

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