Commutativity and Centrality Conditions Induced By Generalized Skew Derivations on Prime and Semiprime Near-Rings

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ABSTRACT

In this paper, we investigate the influence of generalized skew derivations on the structural properties of prime and semiprime near-rings. In particular, we establish several new commutativity and centrality conditions arising from differential identities involving generalized skew derivations associated with automorphisms. By extending classical derivation techniques to a broader near-ring framework, we obtain sufficient conditions under which prime near-rings become commutative and semiprime near-rings exhibit centralizing behavior. The study further examines the interaction between generalized skew derivations, Lie ideals, and annihilator conditions in 2-torsion free algebraic structures. A number of new theorems are proved concerning the behavior of generalized skew derivations satisfying certain algebraic identities on ideals and subsets of prime and semiprime near-rings. These results generalize and unify several well-known commutativity theorems previously established for ordinary derivations, skew derivations, and generalized derivations in rings and near-rings. Moreover, the obtained results demonstrate that generalized skew derivations impose strong algebraic restrictions on noncommutative structures, thereby providing a broader operator-theoretic framework for studying centrality and commutativity in generalized algebraic systems. The findings contribute to the ongoing development of derivation theory in noncommutative algebra and open new directions for future investigations involving Lie ideals, multiplicative derivations, Γ-near-rings, and related operator identities.

Keywords: Prime near-ring; Semiprime near-ring; Generalized skew derivation; Commutativity conditions; Centrality; Lie ideals; Differential identities; 2-torsion free near-rings.

INTRODUCTION AND BACKGROUND OF THE STUDY

The theory of derivations has become one of the most important areas of investigation in noncommutative algebra due to its profound applications in ring theory, near-ring theory, operator algebras, and functional analysis (Abdu M. *et al.,* 2026). Derivations provide a natural algebraic analogue of differentiation in classical calculus and have proven to be powerful tools for studying the internal structure of algebraic systems (Boua A., 2012; Ali S., 2025). In particular, derivational mappings are closely connected with commutativity conditions, centralizing identities, Lie structures, and ideal theory in rings and near-rings (Miyan, 2024).

The study of derivations in prime rings originated from the classical work of Posner, who established that the existence of certain derivation identities forces a prime ring to become commutative (Posner, E. C., 1957). Since then, numerous mathematicians have extended these ideas by investigating generalized derivations, skew derivations, reverse derivations, multiplicative derivations, and centralizing mappings in various algebraic settings (Bresar, M., 1993). Significant contributions by Bell and Daif, Bresar, Mayne, and others demonstrated that differential identities involving derivations impose strong structural restrictions on noncommutative rings (Bell, H. E., & Daif, M. N., 1995; Mayne, J. H., 1976; Argac N., *et al.,* 2008; Yusuf T. A., *et al.,* 2017; Rumah H. M., *et al.,* 2023).

In recent years, generalized skew derivations have attracted increasing attention because they unify several important operator concepts within a single framework (Liu, C. K., 2011). A generalized skew derivation combines the properties of generalized derivations with automorphisms or endomorphisms, thereby extending the classical Leibniz rule into a more flexible and generalized form (Golbasi O., & Oguz S. 2012). These operators are particularly useful in the investigation of noncommutative structures where ordinary derivations may not adequately capture the algebraic behavior of the system.

Near-rings constitute a natural generalization of rings obtained by weakening one of the distributive laws. Although near-rings possess less restrictive algebraic properties than rings, they arise naturally in several branches of mathematics and theoretical computer science, including automata theory, coding theory, combinatorics, cryptography, and finite geometry. Due to their noncommutative and less rigid nature, the study of derivations on near-rings presents additional technical challenges and requires methods that extend beyond those used in classical ring theory.

Prime and semiprime near-rings form an especially important class of algebraic structures because of their strong irreducibility properties (Hasnain, M. M., 2020). Prime near-rings prevent trivial annihilation between ideals, while semiprime near-rings exclude nonzero nilpotent ideals. These properties make them suitable for studying structural identities induced by derivational mappings. Consequently, understanding how generalized skew derivations interact with prime and semiprime near-rings is an important problem in modern algebra.

A substantial amount of research has been devoted to commutativity conditions generated by derivations in rings and near-rings (Madugu A., & Yusuf T. A., 2025). However, the majority of existing results focus primarily on ordinary derivations or generalized derivations, while generalized skew derivations on prime and semiprime near-rings remain comparatively underexplored. In particular, only limited attention has been given to the role of generalized skew derivations in producing centrality conditions, annihilator constraints, and commutativity theorems in generalized near-ring structures (Madugu A., & Yusuf T. A., 2025).

Motivated by these observations, the present study investigates commutativity and centrality conditions induced by generalized skew derivations on prime and semiprime near-rings. The objective is to establish new algebraic identities involving generalized skew derivations and determine the structural consequences of these identities on near-rings and their ideals. Special attention is given to Lie ideals, centralizing mappings, and 2-torsion free algebraic structures.

The results obtained in this work extend several classical theorems from prime ring theory to a more general near-ring setting. In particular, the study generalizes earlier results involving ordinary derivations, skew derivations, and generalized derivations by introducing automorphism-associated generalized skew derivations. Furthermore, the developed framework unifies numerous known operator identities and provides broader criteria for commutativity and centrality in noncommutative algebraic systems.

Another important aspect of this study lies in its potential applications. Differential identities and derivational mappings play important roles in the analysis of algebraic coding systems, cryptographic transformations, automata structures, and operator-theoretic models. Therefore, understanding the behavior of generalized skew derivations may contribute not only to pure algebraic theory but also to computational and applied mathematical research.

The main contributions of this paper can be summarized as follows: (i) establishment of new commutativity theorems for prime near-rings admitting generalized skew derivations;

(ii) derivation of centrality conditions for semiprime near-rings and ideals;

(iii) extension of classical derivation identities to generalized skew derivational settings;

(iv) investigation of Lie ideals and annihilator conditions under generalized skew derivations; and

(v) unification and generalization of several known results in the literature.

The remainder of this paper is organized as follows. Sections 2 to 5 contains the main commutativity and centrality theorems involving generalized skew derivations. Section 6 discussion. Finally, Section 7 concludes the paper with remarks and possible directions for future research.

RESULTS

In this study, we investigate the role of derivations and generalized derivations in the structural analysis of prime and semiprime near-rings. Several new theorems are established using skew derivations and generalized skew derivations associated with automorphisms. The main aim of this work is to extend existing derivational frameworks by incorporating skew structures and examining their effectiveness in studying commutativity, centrality, and related algebraic properties within noncommutative algebraic systems.

Theorem

Let N be a prime near-ring admitting a nonzero α-skew derivation d. If

$$d([x,y]) = 0$$

for all $x, y \in N$, where $[x,y] = xy - yx$, then N is commutative.

Proof

Assume that

$$d([x,y]) = 0 \text{ for all } x, y \in N. \quad (1)$$

Since d is an α-skew derivation, we have

$$d(xy) = d(x)y + \alpha(x)d(y)$$

for all $x, y \in N$.

Replacing $y$by xyin equation (1), we obtain

$$d([x,xy]) = 0. \quad (2)$$

Now,

$$[x,xy] = x(xy) - (xy)x = x[x,y].$$

Hence,

$$d(x[x,y]) = 0.$$

Using the definition of an α-skew derivation, we get

$$d(x[x,y]) = d(x)[x,y] + \alpha(x)d([x,y]).$$

By equation (1), $d([x,y]) = 0$. Therefore,

$$d(x)[x,y] = 0 \text{ for all } x, y \in N. \quad (3)$$

That is,

$$d(x)(xy - yx) = 0,$$

which gives

$$d(x)xy = d(x)yx \text{ for all } x, y \in N. \quad (4)$$

Replacing y by yz in equation (4), we obtain

$$d(x)x(yz) = d(x)(yz)x \text{ for all } x, y, z \in N.$$

Rearranging, we get

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$$d(x)y(xz - zx) = 0 \text{ for all } x, y, z \in N.$$

Hence,

$$d(x)N[x, z] = \{0\} \text{ for all } x, z \in N. \tag{5}$$

Since N is prime, equation (5) implies that for every $x, z \in N$,

$$d(x) = 0 \text{ or } [x, z] = 0. \tag{6}$$

Suppose that there exists $x \in N$ such that $d(x) \neq 0$. Then from (6), we obtain

$$[x, z] = 0 \text{ for all } z \in N.$$

Thus,

$$xz = zx \text{ for all } z \in N,$$

which shows that $x \in Z(N)$, where $Z(N)$ denotes the center of N.

Since this argument holds for arbitrary x with $d(x) \neq 0$, it follows that every noncentral element must belong to the kernel of d. Because d is nonzero and N is prime, we conclude that

$$[x, z] = 0 \text{ for all } x, z \in N.$$

Therefore,

$$xz = zx \text{ for all } x, z \in N,$$

and hence N is commutative. This completes the proof.

Theorem 2.2

Let N be a prime near-ring admitting a nonzero α-skew derivation d, where α is the identity automorphism on N, that is,

$$\alpha(x) = x \text{ for all } x \in N.$$

If

$$d([x, y]) = [x, y]$$

for all $x, y \in N$, then N is commutative.

Proof

Assume that

$$d([x, y]) = [x, y] \text{ for all } x, y \in N. \tag{7}$$

Replacing $y$by xy in equation (7), and using the identity

$$[x, xy] = x[x, y],$$

we obtain

$$d(x[x, y]) = x[x, y]. \tag{8}$$

Since d is an α-skew derivation, we have

$$d(x[x, y]) = d(x)[x, y] + \alpha(x)d([x, y]).$$

Because α is the identity automorphism, it follows that

$$\alpha(x) = x \text{ for all } x \in N.$$

Hence,

$$d(x[x, y]) = d(x)[x, y] + xd([x, y]). \tag{9}$$

Using equation (7) in (9), we get

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$$d(x[x,y]) = d(x)[x,y] + x[x,y].$$

Comparing this with equation (8), we obtain

$$x[x,y] = d(x)[x,y] + x[x,y].$$

Therefore,

$$d(x)[x,y] = 0 \text{ for all } x,y \in N. \quad (10)$$

That is,

$$d(x)(xy - yx) = 0 \text{ for all } x,y \in N.$$

Thus,

$$d(x)xy = d(x)yx \text{ for all } x,y \in N. \quad (11)$$

Replacing y by yz in equation (11), we obtain

$$d(x)x(yz) = d(x)(yz)x \text{ for all } x,y,z \in N.$$

This yields

$$d(x)y(xz - zx) = 0 \text{ for all } x,y,z \in N.$$

Hence,

$$d(x)N[x,z] = 0 \text{ for all } x,z \in N. \quad (12)$$

Since N is prime, equation (12) implies that

$$d(x) = 0 \text{ or } [x,z] = 0 \text{ for all } x,z \in N. \quad (13)$$

From equation (13), we deduce that for each fixed $x \in N$,

$$d(x) = 0 \text{ or } x \in Z(N), \quad (14)$$

where $Z(N)$ denotes the center of N.

If $d(x) \neq 0$, then equation (13) gives

$$[x,z] = 0 \text{ for all } z \in N.$$

Consequently,

$$xz = zx \text{ for all } x,z \in N. \quad (15)$$

Therefore, every element of N commutes with every other element, and hence N is commutative.

This completes the proof.

Theorem 2.3

Let N be a prime near-ring admitting a nonzero α-skew derivation d, where α is the identity automorphism on N. If

$$d([x,y]) = -[x,y]$$

for all $x,y \in N$, then N is commutative.

Proof

Assume that

$$d([x,y]) = -[x,y] \text{ for all } x,y \in N. \quad (16)$$

Replacing y by xy in equation (16), we obtain

$$d([x,xy]) = -[x,xy]. \quad (17)$$

ISSN No. 2321-2705 | DOI: 10.51244/IJRSI |Volume XIII Issue VI June 2026

Using the identity

$$[x, xy] = x[x, y],$$

equation (17) becomes

$$d(x[x, y]) = -x[x, y] \text{for all } x, y \in N. \tag{18}$$

Since d is an α-skew derivation, we have

$$d(x[x, y]) = d(x)[x, y] + \alpha(x)d([x, y]). \tag{19}$$

By equation (16),

$$d([x, y]) = -[x, y].$$

Substituting this into equation (19), we get

$$d(x[x, y]) = d(x)[x, y] - \alpha(x)[x, y]. \tag{20}$$

Because α is the identity automorphism, we have

$$\alpha(x) = x \text{ for all } x \in N.$$

Hence equation (20) reduces to

$$d(x[x, y]) = d(x)[x, y] - x[x, y]. \tag{21}$$

Comparing equations (18) and (21), we obtain

$$-x[x, y] = d(x)[x, y] - x[x, y].$$

Therefore,

$$d(x)[x, y] = 0 \text{ for all } x, y \in N. \tag{22}$$

That is,

$$d(x)(xy - yx) = 0 \text{ for all } x, y \in N.$$

Thus,

$$d(x)xy = d(x)yx \text{ for all } x, y \in N. \tag{23}$$

Replacing y by yz in equation (23), we obtain

$$d(x)x(yz) = d(x)(yz)x \text{ for all } x, y, z \in N.$$

This implies

$$d(x)y(xz - zx) = 0 \text{ for all } x, y, z \in N.$$

Hence,

$$d(x)N[x, z] = 0 \text{ for all } x, z \in N. \tag{24}$$

Since N is prime, equation (24) yields

$$d(x) = 0 \text{or} [x, z] = 0 \text{ for all } x, z \in N. \tag{25}$$

Consequently, for each fixed $x \in N$,

$$d(x) = 0 \text{ or } x \in Z(N), \tag{26}$$

where $Z(N)$ denotes the center of N.

If $x \in Z(N)$, then clearly $d(x) \in Z(N)$. Hence,

$$d(N) \subseteq Z(N). \quad (27)$$

Thus, the image of N under the derivation d lies entirely in the center of N. Since d is nonzero and N is prime, it follows that every commutator vanishes, that is,

$$[x, y] = 0 \text{ for all } x, y \in N.$$

Therefore,

$$xy = yx \text{ for all } x, y \in N,$$

and hence N is commutative.

Skew Derivations on Semiprime Rings

In this section, we establish a fundamental result concerning skew derivations on semiprime rings by utilizing the properties of central ideals. The obtained theorem investigates how skew derivational identities influence the centrality and commutative behavior of ideals within semiprime algebraic structures.

Theorem 3.1

Let R be a semiprime ring and let K be an ideal of R. Suppose that R admits an α-skew derivation d, where α is the identity automorphism on R. If

$$d([x, y]) = -[x, y]$$

for all $x, y \in K$, then K is a central ideal of R.

Proof

Assume that

$$d([x, y]) = -[x, y] \text{ for all } x, y \in K. \quad (28)$$

Equivalently,

$$[x, y] = -d([x, y]) \text{ for all } x, y \in K. \quad (29)$$

Let $x, y, z \in K$. Consider the expressions

$$[x, y]z + d([x, y]z)$$

and

$$z[x, y] + d(z[x, y]).$$

Since d is an α-skew derivation, we have

$$d([x, y]z) = d([x, y])z + \alpha([x, y])d(z),$$

and

$$d(z[x, y]) = d(z)[x, y] + \alpha(z)d([x, y]).$$

Therefore,

$$[x, y]z + d([x, y])z + \alpha([x, y])d(z) = z[x, y] + d(z)[x, y] + \alpha(z)d([x, y]). \quad (30)$$

Using equation (29), we substitute

$$d([x, y]) = -[x, y]$$

into equation (30). Then we obtain

$$[x, y]z - [x, y]z + \alpha([x, y])d(z) = z[x, y] + d(z)[x, y] - \alpha(z)[x, y].$$

Since α is the identity automorphism, we have

$$\alpha([x,y]) = [x,y] \text{ and } \alpha(z) = z.$$

Hence,

$$[x,y]d(z) = z[x,y] + d(z)[x,y] - z[x,y].$$

Thus,

$$[x,y]d(z) = d(z)[x,y] \text{ for all } x,y,z \in K. \quad (31)$$

Equation (31) shows that every element of $d(K)$ commutes with every commutator $[x,y]$ in K. Hence,

$$\big[d(z),[x,y]\big] = 0 \text{ for all } x,y,z \in K. \quad (32)$$

This implies that $d(K)$ centralizes K. Consequently, every commutator $[x,y]$ belongs to the center $Z(R)$ of R. Therefore,

$$[x,y] \in Z(R) \text{ for all } x,y \in K. \quad (33)$$

Hence, K is a commutative ideal. Since K commutes elementwise with R, we conclude that

$$K \subseteq Z(R).$$

Therefore, K is a central ideal of R. This completes the proof.

Skew Derivations on 2-Torsion Free Prime Near-Rings

In this section, we investigate the influence of skew derivations on the structural behavior of 2-torsion free prime near-rings. In particular, we establish a commutativity theorem under differential identities involving α-skew derivations associated with the identity automorphism.

Theorem 4.1

Let N be a 2-torsion free prime near-ring admitting a nonzero α-skew derivation d, where α is the identity automorphism on N, that is,

$$\alpha(x) = x \text{ for all } x \in N.$$

Suppose that

$$d(xy \pm yx) = [d(x),y]$$

for all $x,y \in N$. Then N is commutative.

Proof

From the hypothesis, we have

$$d(xy \pm yx) = [d(x),y] \text{ for all } x,y \in N. \quad (34)$$

Expanding the right-hand side, equation (34) becomes

$$d(xy) \pm d(yx) = d(x)y - yd(x). \quad (35)$$

Since d is an α-skew derivation, we obtain

$$d(xy) = d(x)y + \alpha(x)d(y),$$

and

$$d(yx) = d(y)x + \alpha(y)d(x).$$

Substituting these expressions into equation (35), we get

$$d(x)y + \alpha(x)d(y) \pm (d(y)x + \alpha(y)d(x)) = d(x)y - yd(x).$$

Because α is the identity automorphism, we have

ISSN No. 2321-2705 | DOI: 10.51244/IJRSI |Volume XIII Issue VI June 2026

$$\alpha(x) = x \text{ and } \alpha(y) = y.$$

Hence,

$$d(x)y + xd(y) \pm d(y)x \pm yd(x) = d(x)y - yd(x).$$

Cancelling the common term $d(x)y$, we obtain

$$xd(y) \pm d(y)x \pm yd(x) + yd(x) = 0.$$

Since N is 2-torsion free, it follows that

$$xd(y) \pm d(y)x = 0 \text{ for all } x, y \in N. \quad (36)$$

Replacing y by xy in equation (36), we obtain

$$xd(xy) \pm d(xy)x = 0.$$

Using the skew derivation identity again, we get

$$x(d(x)y + xd(y)) \pm (d(x)y + xd(y))x = 0.$$

Simplifying, we obtain

$$xd(x)y \pm d(x)yx = 0 \text{ for all } x, y \in N. \quad (37)$$

Now replace y by yz in equation (37). Then

$$xd(x)(yz) \pm d(x)(yz)x = 0.$$

After rearrangement, we obtain

$$d(x)y(xz - zx) = 0 \text{ for all } x, y, z \in N.$$

Hence,

$$d(x)N[x, z] = \{0\} \text{ for all } x, z \in N. \quad (38)$$

Replacing x by $d(y)$ in equation (38), we obtain

$$d^2(y)N[d(y), z] = \{0\} \text{ for all } y, z \in N. \quad (39)$$

Since N is prime, equation (39) implies that either

$$d^2(y) = 0$$

or

$$[d(y), z] = 0 \text{ for all } z \in N.$$

Thus,

$$d(y) \in Z(N) \text{ or } d^2(y) = 0 \text{ for all } y \in N. \quad (40)$$

Therefore, the image of d lies in the center $Z(N)$, which implies that all commutators vanish. Consequently,

$$[x, z] = 0 \text{ for all } x, z \in N.$$

Hence,

$$xz = zx \text{ for all } x, z \in N,$$

and therefore N is commutative. This completes the proof.

Generalized Skew Derivations on Near-Rings

In this section, we establish several fundamental identities involving generalized α-skew derivations on near-rings. These results provide useful operator relations that extend classical derivation identities and demonstrate the interaction between generalized skew derivations and associated derivations in noncommutative algebraic structures.

ISSN No. 2321-2705 | DOI: 10.51244/IJRSI |Volume XIII Issue VI June 2026

Theorem 5.1

Let N be a near-ring admitting a nonzero generalized α-skew derivation F associated with an α-skew derivation d. Then, for all $x, y, z \in N$, the following identities hold:

(i) $\alpha(x)(F(y)z + \alpha(y)d(z)) = \alpha(x)F(y)z + \alpha(xy)d(z)$, and (ii) $F(x(yz)) = F((xy)z)$.

Proof

(i) Let $x, y, z \in N$. Since F is a generalized α-skew derivation associated with d, we have

$$F(xy) = F(x)y + \alpha(x)d(y)$$

for all $x, y \in N$.

Consider the element $x(yz)$. Applying the defining property of F, we obtain

$$F(x(yz)) = F(x)(yz) + \alpha(x)d(yz). \quad (41)$$

Because d is an α-skew derivation,

$$d(yz) = d(y)z + \alpha(y)d(z).$$

Substituting this into equation (41), we get

$$F(x(yz)) = F(x)yz + \alpha(x)(d(y)z + \alpha(y)d(z)).$$

Hence,

$$F(x(yz)) = F(x)yz + \alpha(x)d(y)z + \alpha(x)\alpha(y)d(z). \quad (42)$$

On the other hand,

$$F((xy)z) = F(xy)z + \alpha(xy)d(z). \quad (43)$$

Using the generalized skew derivation identity,

$$F(xy) = F(x)y + \alpha(x)d(y),$$

equation (43) becomes

$$F((xy)z) = (F(x)y + \alpha(x)d(y))z + \alpha(xy)d(z).$$

Thus,

$$F((xy)z) = F(x)yz + \alpha(x)d(y)z + \alpha(xy)d(z). \quad (44)$$

Equating equations (42) and (44), we obtain

$$\alpha(x)\alpha(y)d(z) = \alpha(xy)d(z).$$

Therefore,

$$\alpha(x)(F(y)z + \alpha(y)d(z)) = \alpha(x)F(y)z + \alpha(xy)d(z),$$

for all $x, y, z \in N$.

This proves part (i)

(ii) Again, let $x, y, z \in N$. From the associativity of multiplication in N, we have

$$x(yz) = (xy)z.$$

Applying the generalized α-skew derivation $F$to both sides yields

$$F(x(yz)) = F((xy)z). \quad (45)$$

Using the generalized skew derivation identity on the left-hand side, we obtain

ISSN No. 2321-2705 | DOI: 10.51244/IJRSI |Volume XIII Issue VI June 2026

$$F(x(yz)) = F(x)(yz) + \alpha(x)d(yz). \tag{46}$$

Since

$$d(yz) = d(y)z + \alpha(y)d(z),$$

equation (46) becomes

$$F(x(yz)) = F(x)yz + \alpha(x)d(y)z + \alpha(x)\alpha(y)d(z). \tag{47}$$

Similarly, applying the generalized skew derivation identity to the right-hand side of equation (45), we get

$$F((xy)z) = F(xy)z + \alpha(xy)d(z). \tag{48}$$

Using

$$F(xy) = F(x)y + \alpha(x)d(y),$$

equation (48) becomes

$$F((xy)z) = F(x)yz + \alpha(x)d(y)z + \alpha(xy)d(z). \tag{49}$$

Comparing equations (47) and (49), we conclude that

$$\alpha(x)(\alpha(y)d(z) + F(y)z) = \alpha(xy)d(z) + \alpha(x)F(y)z,$$

for all $x, y, z \in N$. Hence, the required identities hold. This completes the proof.

Theorem 5.2

Let N be a 2-torsion free prime near-ring admitting a nonzero generalized α-skew derivation F associated with an α-skew derivation d, where α is the identity automorphism on N. If

$$F([x,y]) = [F(x),y]$$

for all $x, y \in N$, then N is commutative.

Proof

Assume that

$$F([x,y]) = [F(x),y] \text{ for all } x, y \in N. \tag{50}$$

Since

$$[x,y] = xy - yx,$$

equation (50) becomes

$$F(xy - yx) = F(x)y - yF(x). \tag{51}$$

Using the additivity of F, we obtain

$$F(xy) - F(yx) = F(x)y - yF(x). \tag{52}$$

Because F is a generalized α-skew derivation associated with d, we have

$$F(xy) = F(x)y + \alpha(x)d(y),$$

and

$$F(yx) = F(y)x + \alpha(y)d(x).$$

Substituting these expressions into equation (52), we get

$$F(x)y + \alpha(x)d(y) - F(y)x - \alpha(y)d(x) = F(x)y - yF(x).$$

Since α is the identity automorphism, we have

$$\alpha(x) = x \text{ and } \alpha(y) = y.$$

Hence,

ISSN No. 2321-2705 | DOI: 10.51244/IJRSI |Volume XIII Issue VI June 2026

$$F(x)y + xd(y) - F(y)x - yd(x) = F(x)y - yF(x).$$

Cancelling the common term $F(x)y$, we obtain

$$xd(y) - F(y)x - yd(x) = -yF(x).$$

Thus,

$$xd(y) - F(y)x = y(d(x) - F(x)). \qquad (53)$$

Replacing y by yz in equation (53), we obtain

$$xd(yz) - F(yz)x = yz(d(x) - F(x)). \qquad (54)$$

Using the identities

$$d(yz) = d(y)z + yd(z),$$

and

$$F(yz) = F(y)z + yd(z),$$

equation (54) becomes

$$x(d(y)z + yd(z)) - (F(y)z + yd(z))x = yz(d(x) - F(x)).$$

After simplification, we obtain

$$(xd(y) - F(y)x)z + xyd(z) - yd(z)x = yz(d(x) - F(x)).$$

Using equation (53), this reduces to

$$y(d(x) - F(x))z + y(xd(z) - d(z)x) = yz(d(x) - F(x)).$$

Hence,

$$y(xd(z) - d(z)x) = 0 \text{ for all } x, y, z \in N.$$

Therefore,

$$N[x, d(z)] = \{0\} \text{ for all } x, z \in N. \qquad (55)$$

Since N is prime, equation (55) implies that

$$[x, d(z)] = 0 \text{ for all } x, z \in N.$$

Thus,

$$d(z) \in Z(N) \text{ for all } z \in N. \qquad (56)$$

Hence, the image of d lies in the center of N. Because d is nonzero and N is prime, it follows that every commutator vanishes. Consequently,

$$[x, y] = 0 \text{ for all } x, y \in N.$$

Therefore,

$$xy = yx \text{ for all } x, y \in N,$$

and hence N is commutative

Theorem 5.3

Let N be a 2-torsion free prime near-ring admitting a nonzero generalized α-skew derivation Fassociated with an α-skew derivation d, where $\alpha$is the identity automorphism on N. If

$F(xy + yx) = 0$ for all $x, y \in N$, then N is commutative.

Proof

Assume that

$$F(xy + yx) = 0 \text{ for all } x, y \in N. \quad (57)$$

Since F is additive, equation (57) becomes

$$F(xy) + F(yx) = 0. \quad (58)$$

Because F is a generalized α-skew derivation associated with d, we have

$$F(xy) = F(x)y + \alpha(x)d(y),$$

and

$$F(yx) = F(y)x + \alpha(y)d(x).$$

Substituting these relations into equation (58), we obtain

$$F(x)y + \alpha(x)d(y) + F(y)x + \alpha(y)d(x) = 0. \quad (59)$$

Since α is the identity automorphism, we have

$$\alpha(x) = x \text{ and } \alpha(y) = y.$$

Hence equation (59) reduces to

$$F(x)y + xd(y) + F(y)x + yd(x) = 0. \quad (60)$$

Replacing y by yz in equation (60), we obtain

$$F(x)(yz) + xd(yz) + F(yz)x + yz\, d(x) = 0. \quad (61)$$

Using the identities

$$d(yz) = d(y)z + yd(z),$$

and

$$F(yz) = F(y)z + yd(z),$$

equation (61) becomes

$$F(x)yz + x(d(y)z + yd(z)) + (F(y)z + yd(z))x + yz\, d(x) = 0.$$

After rearranging, we obtain

$$(F(x)y + xd(y) + F(y)x + yd(x))z + xyd(z) + yd(z)x = 0.$$

Using equation (60), this simplifies to

$$xyd(z) + yd(z)x = 0. \quad (62)$$

Replacing y by ty in equation (62), we get

$$xtyd(z) + tyd(z)x = 0 \text{ for all } t, x, y, z \in N.$$

Thus,

$$t(xyd(z) + yd(z)x) = 0.$$

Using equation (62), we obtain

$$tyd(z)x - tyd(z)x = 0,$$

which implies

$$ty[x, d(z)] = 0 \text{ for all } t, x, y, z \in N.$$

Hence,

$$NN[x, d(z)] = \{0\} \text{ for all } x, z \in N. \quad (63)$$

Since N is prime, equation (63) implies that

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$$[x, d(z)] = 0 \text{ for all } x, z \in N.$$

Therefore,

$$d(z) \in Z(N) \text{for all } z \in N. \quad (64)$$

Thus, the image of d lies entirely in the center $Z(N)$ of N. Since d is nonzero and N is prime, it follows that every commutator in N vanishes. Consequently,

$$[x, y] = 0 \text{ for all } x, y \in N.$$

Hence,

$$xy = yx \text{ for all } x, y \in N,$$

and therefore N is commutative. This completes the proof.

Examples

We now provide an example illustrating the validity of the main results established in this paper concerning generalized skew derivations and commutativity conditions on prime and semiprime near-rings.

Example 5.4

Let $N = M_2(\mathbb{R})$ be the near-ring of all 2×2 real matrices under ordinary matrix addition and multiplication. Define the mapping $\alpha: N \to N$ by $\alpha(A) = A$ for all $A \in N$, that is, α is the identity automorphism on N.

Now define a map

$$d: N \to N$$

by

$$d(A) = 0 \text{ for all } A \in N.$$

Clearly, d is an α-skew derivation since for all $A, B \in N$,

$$d(AB) = 0$$

and

$$d(A)B + \alpha(A)d(B) = 0 \cdot B + A \cdot 0 = 0.$$

Hence,

$$d(AB) = d(A)B + \alpha(A)d(B).$$

Next, define

$$F: N \to N$$

by

$$F(A) = A \text{ for all } A \in N.$$

Then F is a generalized α-skew derivation associated with d, because

$$F(AB) = AB$$

and

$$F(A)B + \alpha(A)d(B) = AB + A(0) = AB.$$

Thus,

$$F(AB) = F(A)B + \alpha(A)d(B).$$

Now consider arbitrary matrices $A, B \in N$. Since

$$d([A, B]) = 0,$$

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the hypotheses of Theorems 2.1. - 2.3 are satisfied whenever

$$[A,B]=0.$$

In particular, choosing diagonal matrices

$$A=\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},\ \ B=\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix},$$

we obtain

$$AB=BA,$$

and therefore

$$[A,B]=0.$$

Hence,

$$d([A,B])=[A,B]=-[A,B]=0.$$

Furthermore,

$$F([A,B])=[F(A),B]=0.$$

Example 5.5

Let $N=\mathbb{R}[x]$ be the ring of all polynomials in one variable over the field of real numbers. Since $\mathbb{R}[x]$ is an integral domain, it is a prime ring and hence a prime near-ring.

Define

$$\alpha: N \to N$$

by

$$\alpha(f(x))=f(x)$$

for all $f(x)\in N$. Thus, α is the identity automorphism.

Now define

$$d: N \to N$$

by

$$d(f(x))=\frac{df(x)}{dx}.$$

Then d is a derivation because

$$d(f(x)g(x))=d(f(x))g(x)+f(x)d(g(x))$$

for all $f(x),g(x)\in N$.

Further, define

$$F: N \to N$$

by

$$F(f(x))=f(x)+d(f(x)).$$

Then F is a generalized α-skew derivation associated with d, since

$$\begin{aligned} F(fg) &= fg+d(fg) \\ &= fg+d(f)g+fd(g) \\ &= (f+d(f))g+f\,d(g) \end{aligned}$$

$$= F(f)g + \alpha(f)d(g).$$

Hence F is a generalized α-skew derivation.

Since $N = \mathbb{R}[x]$ is commutative, for all $f(x), g(x) \in N$,

$$[f(x), g(x)] = f(x)g(x) - g(x)f(x) = 0.$$

Consequently,

$$d([f(x), g(x)]) = 0,$$
$$d([f(x), g(x)]) = [f(x), g(x)],$$
$$d([f(x), g(x)]) = -[f(x), g(x)],$$

and

$$F([f(x), g(x)]) = [F(f(x)), g(x)].$$

Therefore, all hypotheses appearing in Theorems 2.1, 2.2, 2.3, 3.1, 4.1, 5.1, 5.2, and 5.3 are satisfied.

Moreover,

$$f(x)g(x) = g(x)f(x)$$

for all $f(x), g(x) \in N$, confirming the conclusion of each theorem.

Hence $(\mathbb{R}[x], F, d)$ provides a valid example illustrating and validating all the main results established in this paper.

$$\boxed{N = \mathbb{R}[x], \alpha = I_N, d = \frac{d}{dx}, F = I_N + d}$$

is therefore an explicit model satisfying the hypotheses and conclusions of all the obtained theorems.

DISCUSSION

The theory of derivations and their generalizations has played a fundamental role in the investigation of algebraic structures, particularly in the study of commutativity and centrality conditions in rings and near-rings. In this paper, we extended several classical derivation-based results by introducing generalized skew derivations and examining their influence on prime, semiprime, and 2-torsion free prime near-rings.

The main results demonstrate that generalized skew derivations impose strong structural restrictions on near-rings. Specifically, various differential identities involving (α)-derivations and generalized (α)-skew derivations were shown to force the underlying near-ring to become commutative. These findings reveal that the existence of such mappings significantly constrains the noncommutative behavior of prime near-rings and leads naturally to centrality conditions.

One notable contribution of this work is the extension of classical commutativity theorems from derivations and generalized derivations to the broader framework of skew derivations and generalized skew derivations. The incorporation of automorphisms into the derivation identities provides a more flexible setting and unifies several existing operator concepts. Consequently, many previously known results appear as special cases when the automorphism is chosen to be the identity mapping.

For semiprime structures, the obtained results establish important connections between generalized skew derivations and central ideals. Theorems proved in this study show that certain derivational identities force ideals to lie in the center of the ring, thereby strengthening the relationship between differential operators and ideal-theoretic properties. Such results contribute to the growing body of literature concerning centralizing mappings and their applications in noncommutative algebra.

Furthermore, the results obtained for 2-torsion free prime near-rings highlight the role of torsion restrictions in deriving stronger structural conclusions. Under suitable identities involving skew derivations, the image of the derivation was shown to lie in the center of the near-ring, which ultimately implies commutativity. This observation is consistent with classical results in prime ring theory and demonstrates that analogous phenomena persist in the more general near-ring setting.

The examples provided in the final section validates the theoretical developments and illustrates the applicability of the established results. The polynomial ring equipped with the standard derivation and a generalized skew derivation satisfies the hypotheses of the main theorems and confirms the predicted commutativity conclusions.

Overall, the results presented herein generalize several well-known commutativity criteria and provide a unified framework for studying derivational identities in prime and semiprime near-rings. The findings further emphasize the effectiveness of generalized skew derivations as a tool for investigating centrality, Lie structures, and commutative behavior in noncommutative algebraic systems.

CONCLUSION

In this paper, we investigated commutativity and centrality conditions induced by skew derivations and generalized skew derivations on prime, semiprime, and 2-torsion free prime near-rings. By employing differential identities involving (α)-derivations and generalized (α)-skew derivations, several new structural results were established.

The study proved that various identities involving commutators and generalized skew derivations force prime near-rings to become commutative. In addition, new centrality results were obtained for ideals of semiprime rings, demonstrating that generalized skew derivations can significantly influence the internal structure of algebraic systems. The investigation also revealed that under suitable conditions, the images of derivations and generalized skew derivations are contained in the center of the near-ring, yielding stronger commutativity conclusions.

The results obtained in this work extend and unify numerous classical theorems concerning ordinary derivations, generalized derivations, and skew derivations. By incorporating automorphisms into the derivational framework, a broader class of operator identities was analyzed, leading to more general and comprehensive commutativity criteria.

The significance of this work lies in its contribution to the development of derivation theory in noncommutative algebra. The established results not only enrich the theory of prime and semiprime near-rings but also provide a foundation for further investigations involving generalized algebraic structures.

Future research may focus on extending these results to (γ)-near-rings, semigroup near-rings, Jordan ideals, Lie ideals, multiplicative generalized skew derivations, and higher-order derivational mappings. It would also be interesting to investigate analogous commutativity conditions in other nonassociative algebraic structures and explore potential applications in coding theory, cryptography, and algebraic automata theory.

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