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Mathematical Model of a Degenerate-Pension Wealth Generation Strategies, with Constant Interest Rate: The Ito' Product Law Approach

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ABSTRACT

This research seeks to develop completely new formulations for portfolio management strategies, for Degenerate-pension wealth, in a DC Pension scheme, with constant interest rate, during the wealth accumulation stage. The Pension plan member (PPM) invested money in savings account (a risk-free asset), in mutual benefit account; the Nigerian Stanbic IBTC Bank's Money market (a risk-less asset), and in Stock (a risky asset), under the Geometric Brownian Motion (GBM) model. Using the Ito' Product Law, an Ordinary Stochastic Differential Equation, representing the evolution of the Degenerate-Pension wealth optimization program was developed. Thereafter, a nonlinear Partial Differential Equation was obtained, using the associated Hamilton Jacobi Bellman (H.J.B) equation, for the optimality condition. The explicit solution of the constant relative risk aversion (CRRA) utility function was obtained, using Legendre transform, dual theory, and change of variable methods. It was established herein that the annuity term due to the chosen utility function vanishes, which depicts a sharp collapse in investment in risky assets. Theorem is constructed and proved on the Degenerate-pension wealth generation strategies.

Keywords: Inflation; Portfolio; Degenerate-Pension Wealth; CARA; Ito' Product Law. MSC: 91G80; 60H10; 93E20

INTRODUCTION

Pension is an economic security [3]. Roughly speaking, according to [3], Pension fund is a specific pull fund into which a sum of money is paid during an employees' active years, which is in turn used to service his/her periodic payments (pension), after retirement. There are two major designs of pension scheme, namely, the Defined Benefit (DB) pension scheme, and the Defined Contribution (DC) pension scheme. As the names implies, in the DB scheme, the benefits of the plan member are defined, and the sponsor (the employee) bears the financial risk. Whereas, in the DC pension scheme, the contributions are defined, the retirement benefits depend on the contributions and the investment returns, and the contributors (the pension plan members) bear the financial risk [9]. Due to the fact that the problem of funding is taken care of through the compulsory contributions by the employees, which in turn made the scheme to be fully funded, the DC pension scheme became dominant [9]. Again, since how appreciated the pension wealth is, is a function of the level of strategic investments made with the pension fund, hence the need to seek the best possible investment strategies became very necessary.

Since the problem of utility maximization is of great importance [14], hence the reason many researchers have made several contributions in this direction. To mention but a few; Cairns [4] worked on stochastic life styling: optimal dynamic asset allocation for DC pension scheme. Therein, various properties and characteristics of the optimal asset allocation strategy, both with and without the presence of non-hedge able salary risk were discussed. More so, the significance of alternative optimal strategy by pension managers was established. Gao [6] did a work on "optimal investment strategy for annuity contracts, under the constant elasticity of variance (CEV) model". In his work made an assertion without a proof that the elastic parameter is not equal to 1. More so, the work did not consider in its pension wealth model the fact that the next-of-kin of the dead contributors should be paid some benefits, neither did it consider categories of contributors, since in every employment period, a certain group of people of the same cadre is employed, therefore, in real life situation, the right thing should be multiple and categorized contributors. It did not also indicate that contributors will not willingly withdraw from the scheme, hence the modification by [9]. Therein, a proposition was proved on the condition where the elastic parameter will not be equal to 1 and also considered various categories of contributors. This their work was later reviewed and generalized in [3]. [5] took a different direction when they investigated optimal investment strategy for a DC pension fund with a stochastic salary, under the affine interest rate model. In their work, they introduced the notion of "Relative Pension Wealth", that is, where a pension plan member only considers his/her post-retirement benefit, the ratio of the pension wealth to his terminal salary, and this triggered the interest of K. N. C in [11]. In their work, they introduced the notion of extra contribution. It was observed therein that the pension plan member will increase the proportion of his/her wealth to be invested in stock and bond, and reduce that of cash. Othusitse [13] did a work under the inflationary market, with minimum guarantee. In their work, the plan member amortized the pension fund. The CRRA utility function was used to maximize the terminal wealth. This triggered the work done by Kevin N. C. [3]. The investigation reveals that the inflation has significant negative effect on optimal investment strategy, particularly, the RRA is not constant with the investment strategy since the inflation parameters and coefficient of CRRA utility function have insignificant input on the investment strategy.

Mwanakatwe et al [10] analysed the optimal investment strategies for a DC pension fund under the Hull-White interest rate model. Under this model, the pension fund manager can invest capital in the bank account, stock index, and real estates. More so, Battocchio et al [2] studied optimal pension management in a stochastic framework, they came out with a significant result.

Summary of the Literature and Knowledge Gap

It is observed that all the works, done so far in the literature is based on optimal wealth investment strategies on either pension wealth, relative wealth, or residual (surplus) wealth. No work is done yet, applying the well-known Ito' Product Law. Again, no work has been done on the optimization of a Degenerate-Pension Wealth, due to the product of the pension wealth process and the price of inflation, rather, in the literature, Inflation-indexed Bond and Inflation-linked Stock has been considered.

Motivation of Study

To apply the Ito' Product Law in the modelling of the evolution of the two-dimensional Degenerate-Pension wealth process (i.e., Pension wealth that is affected by Inflation), with constant interest rate, for annuity contracts. This situation of constant interest rate was due to the constant remittances of interest into PPM's RSA, at the Imo State University, Owerri, Imo State, Nigeria.

My Interest

To formulate Degenerate-pension wealth generation strategies, expose the extent of damage that is caused by the Inflation and constant interest rate on the Pension wealth, to enable us properly advice the users of the result of this research.

Preliminaries

Let a complete and frictionless financial market that is continuously open over the fixed time interval $[0, \mathrm{T}]$, for $\mathrm{T} > 0$, plan member's terminal time be respectively defined.

Suppose that the market is composed of a risk-free asset (savings account), a risk-less asset (mutual benefit account) and risky asset (stock). Let $(\Omega, \mathrm{F}, \mathrm{P})$ be a complete probability space, where Ω is a real space and P is a probability measure, $\{w_s(t), w_t(t)\}$ are two standard orthogonal Brownian motions, and other orthogonal Brownian motions are

$\{w_s(t), w_r(t)\}, \{w_r(t), w_I(t)\}, \{w_s(t), w_I(t)\}, \{w_I(t), w_t(t)\}, \text{and } \{w_s(t), w_r(t)\}$, $\{F_t(t), F_s(t)\}$ are right continuous

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filtrations whose information are generated by the two standard Brownian motions $\{w_s(t), w_t(t)\}$ and

$\{w_s(t), w_r(t)\}, \{w_r(t), w_I(t)\}, \{w_s(t), w_I(t)\}, \{w_I(t), w_t(t)\}, and\ \{w_s(t), w_r(t)\}$, whose sources of uncertainties are respectively to the stock market, Inflation, interest rate and time evolution.

METHODOLOGY

The following algorithm and tools were used to obtain our results in this research

Hamilton-Jacobi-Bellman (HJB) equation

Define $u_t = (u_s(t), u_B(t))$ as the strategy and the utility that was attained by the investor from a given state n at time t by

$$H_u(t,r,n) = E_u\left[U(N(t)) : r_R(t) = r, n(t) = n\right] \quad (2.1.1)$$

where t is the time, r_R is the short interest rate and n is the Degenerate-wealth. Here, we required to obtain the optimal value function, H, defined by

$$H(t,r_R,n) = Sup_{u_t} H_{u_t}(t,r_R,n) \quad (2.1.2)$$

and the optimal strategy

$$u_t^* = (u_s^*(t), u_B^*(t)) \quad (2.1.3)$$

such that

$$H_{u_t^*}(t,r,y) = H(t,r_R,n) \quad (2.1.4)$$

Legendre Transformation

The well-known Legendre transform and dual theory were used to transform the nonlinear partial differential equation, formed due to sub-section (2.1), to a linear partial differential equation.

Theorem 2.2 (Jonsson and Sircar [8], Chubing Zhang et al [5]): Let f: $R^n \to R$ be a convex function for $z > 0$, then the Legendre transform is defined as;

$$L(z) = \max_n \{f(n) = zn\} \quad (2.2.1)$$

where L(z) is the Legendre dual of $f(n)$.

Now, since $f(n)$ is convex, from theorem 2.2 we defined the Legendre transform, thus

$$\hat{H}(t,r_R,z) = Sup\{H(t,r_R,n) - zn : 0 < n < \infty\},\ 0 < t < T \quad (2.2.2)$$

given that $\hat{H}$ is the dual of H and $z > 0$ is the dual variable of n.

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It is important to note that f and $\breve{H}$ are closely related and as such, called the dual of H

Consequently, we have that

$$\breve{H}\left(t, r_R, z\right) = H\left(t, r_R, z\right) - zf \qquad \text{(Chubing Zhang et al, [5])} \qquad (2.2.4)$$

where,

$$f\left(t, r_R, z\right) = n, H_n = z, f = -\breve{H}_z. \qquad (2.2.5)$$

Similar to equation (2.2.2) and (2.2.3), we have at terminal time, T that

$$\hat{U}\left(z\right) = Sup\left\{U\left(n\right) - zn : 0 < n < T\right\} \qquad (2.2.6)$$

and

$$H\left(z\right) = Sup\left\{n : \mathrm{U}\left(n\right) \geq zn + \hat{U}\left(z\right)\right\}. \qquad (2.2.7)$$

Consequently, $H\left(z\right) = U^{-1}\left(z\right)$ (see Chubing Zhang et al [5]) (2.2.7a)

where H is the inverse of the marginal utility U and note that $H\left(T, r_R, n\right) = U\left(n\right)$.

Consequently, at terminal time T, we have that equation (2.2.2) and (2.2.3) becomes

$$f\left(T, r_R, n\right) = \inf_{n>0}\left\{n : U\left(n\right) \geq zn + \breve{H}\left(\mathrm{t}, r_R, z\right)\right\} \qquad (2.2.8)$$

and

$$\breve{H}\left(\mathrm{t}, r_R, z\right) = Sup_{n>0}\left\{U\left(n\right) - zn\right\} \qquad (2.2.9)$$

Consequently, following equation (2.2.7a), $f\left(\mathrm{T}, r_R, z\right) = U^{-1}\left(z\right)$. (2.2.10)

The Optimization Model

This session introduces the financial market, with assets of interest, and proposes the Degenerate-pension wealth model, in the generation stage.

The Financial Market

Here, we consider a financial market that consists of a risk-free asset (i.e., savings account), a riskless asset (mutual benefit account) and a risky asset (stock).

Define the price of the risky asset by S_t, for any time, $t > 0$, where it evolve, not limited to as in Gao [7], Akpanibah et al [1], and Njoku et [9] thus;

$$dS_t = \mu_s S_t dt + \delta_s S_t dW_s; \quad S_{t_0} = S_0 \qquad (3.1.3)$$

where $\mu_s\left(\mu_s > r_R\right)$ is the expected rate of return on stock, δ_s is a constant volatility on stock.

Define the price of Inflation by $I(t)$, where it evolves thus

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$$dI(t) = \pi_e I(t)dt + \delta_I I(t)dW_I; \quad I(0) = i \tag{3.1.4}$$

where, δ_I is a constant volatility scale factor, due to inflation, π_e is the expected rate of inflation.

Let $\{W_t, W_s, W_I; t, s, i \geq 0\}$ denote a standard Brownian motion, defined on a probability space, $(\Omega, \mathrm{F}, \mathrm{P})$ where $F = \{F_t, F_s, F_i\}$ is an augmented filtration generated by the Brownian motion.

Real Life Model Assumption

Consistent with the Nigerian Pension Reform Act of 2006, as amended [12], we make the following assumptions

(a) The Pension Scheme accumulates wealth
(b) There is only one contributor
(c) The investor invests his/her money in Mutual funds (StanbicIBTC money market) and Stock
(d) Both the Mutual benefit and the stock evolves, following GBM model
(e) The contributor will not willingly withdraw from the scheme
(f) There is no operating cost for the traded equity, hence the trading companies are large corporations.
(g) The expected rate of return on stock is constant due to pre-determined returns by the corporation (Stock Brockers) (case study of Imo State University, Owerri,Imo State, Nigeria)
(h) The trading economy is turbulent, due to inflation.
(i) Complete Orthogonal relationship is considered between all random variables.
(j) The salary of the contributor is unit

Model Formation of the Pension Wealth

Here, the stochastic differential equation that represents the evolution of the pension wealth generation process is constructed as in Chubing Zhang et al [5].

Let $u_m(t), u_B(t)\, and\, u_s(t)$ denote the proportion of the pension fund, put in Savings account, Mutual Benefit account and Stock, respectively.

The associated pension wealth process structure as in Gao [7], Chubing Zhang et al [5], Njoku [9 , 3] is given by

$$dW(t) = (1 - u_B - u_s)W(t)\frac{dM(t)}{M(t)} + u_B W(t)\frac{dB(t)}{B(t)} + u_s W(t)\frac{dS(t)}{S(t)} + cdt \tag{3.3.1}$$

$$\begin{aligned} s.t \quad & u_B(t) > 0 \\ & u_s(t) > 0 \\ & (1 - u_B - u_s) > 0 \end{aligned} \tag{3.3.1a}$$

Inserting equations (3.1.1), (3.1.2), (3.1.3) into (3.3.1) s.t (3.3.1a)

$$dW(t) = W(t)\{[(1 - u_B - u_s)r_R + \mu u_B + \mu_s u_s]dt + [u_B \pi\theta dW_t + u_s \delta_s dW_s]\} \tag{3.3.2}$$

$$\begin{aligned} s.t \quad & u_B(t) > 0 \\ & u_s(t) > 0 \\ & (1 - u_B - u_s) > 0 \end{aligned} \tag{3.3.2a}$$

Model Formation of the Degenerate-Pension Wealth

Here, we applied the well-known Ito' Product Law, to model a stochastic differential equation, which takes into account a unique relationship between the Pension wealth process, the price of Inflation, and equation (3.3.2) s.t (3.3.2a)

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In order to achieve this, we define a new state random variable, $N(t)=I(t)W(t)$, called the Degenerate-Pension Wealth process.

Combining equation (3.1.4), (3.3.2) s.t (3.3.2a) and applying Ito' property (Ito' lemma) and formula, we resolve the stochastic differential equation for $N(t)$ as

$$dN(t)=N(t)\left\{\pi_e dt+\delta_I dw_I+\left[(1-u_B-u_s)r_R+\mu u_B+\mu_s u_s\right]dt+u_B\pi\theta dw_t+u_s\delta_s dw_s\right\} \quad (3.4.1)$$

$$\begin{aligned} s.t \quad & u_B(t)>0 \\ & u_s(t)>0 \\ & (1-u_B-u_s)>0 \\ & N(t)=I(t)W(t)>0 \end{aligned} \quad (3.4.1a)$$

The associated H.J.B to the evolution of the Degenerate-Pension Wealth Process

Under this sub-section, we used the H.J.B equation to maximize equation (3.4.1) s.t (3.4.1a)

Applying the associated H.J.B equation as in Gao [7], Zhang et al [5], and Njoku et al [9, 3] on equation (3.4.1) s.t (3.4.1a), we have

(3.5.1)

$$\text{Subject to;} \quad \begin{aligned} s.t \quad & u_B(t)>0 \\ & u_s(t)>0 \\ & (1-u_B-u_s)>0 \\ & N(t)=I(t)W(t)>0 \\ & r_R = cons\tan t \end{aligned} \quad (3.5.1a)$$

To obtain the optimal value, $u_t^*(t)=(u_B^*(t),u_s^*(t))$, we differentiate equation (3.5.1) s.t (3.5.1a), with respect to u_B *and* u_s, and we have

$$u_B^*=\frac{-H_N(\mu-r_R)}{H_{NN}\pi^2\theta^2N(t)}. \quad (3.5.1b)$$

$$u_S^*=\frac{-H_N(\mu_s-r_R)}{H_{NN}\delta_s^2N(t)} \quad (3.5.1c)$$

Remark 1

Observe that δ_I, which is the constant volatility scale factor, due to inflation, and π_e, which is the expected rate of inflation, are jointly absent in the equations (3.5.1b) and (3.5.1c)

Consequently, they are designated, thus

$$H_t+H_N\{N(t)[\pi_e+(1-u_B-u_s)r_R+\mu u_B+\mu_s u_s]+\frac{1}{2}H_{NN}[\delta_I^2+u_B^2\pi^2\theta^2+u_s^2\delta_s^2]N^2(t)=0$$

$$\delta_I,\pi_e=0$$

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Substituting u_B^* and u_S^* for u_B and u_S in equation (3.5.1) s.t (3.5.1a) and considering $\delta_I, \pi_e = 0$

$$H_t + N(t) r_R H_N + \frac{1}{2}\delta_I^2 N^2(t) H_{NN} + \left[r_R \left(\frac{\mu - r_R}{\pi^2\theta^2} + \frac{\mu_S - r_R}{\delta_S^2} \right) - \frac{\mu(\mu - r_R)}{\pi^2\theta^2} - \frac{\mu_S(\mu_S - r_R)}{\delta_S^2} + \frac{\mu - r}{2\pi^2\theta^2} + \frac{(\mu_S - r_R)}{\delta_S^2} \right] \frac{H_N^2}{H_{NN}} = 0$$

(3.5.2)

Subject to
$$\begin{aligned} & u_B(t) > 0 \\ & u_s(t) > 0 \\ & (1 - u_B - u_s) > 0 \\ & N(t) = I(t) W(t) > 0 \\ & r_R = cons \tan t \end{aligned}$$
(3.5.2a)

Having converted the stochastic control problem described in the previous session to a nonlinear partial differential equation, the next tax is to solve for H in equation (3.5.2) s.t (3.5.2a) and subsequently substitute it into (3.5.1b) and (3.5.1c), to enable us obtain the optimal Degenerate-Pension wealth generation strategies (i.e., the control strategies). For us to achieve this, we apply the well-known Dual theory and Legendre transformation methods, respectively.

THE DUAL AND LEGENDRE TRANSFORM ON EQUATION (3.5.2) S.T (3.5.2A)

Here, we transform the nonlinear second order stochastic partial differential equation (3.5.2) s.t (3.5.2a) into a linear PDE, using the associated Dual theory and Legendre transformations as in Gao [7], Chubing Zhang et al [5], Njoku et al [9, 11], that is;

$$\phi_y = z \text{ and } H_n = z, H_t = \breve{H}_t, H_r = \breve{H}_r, H_{rr} = \breve{H}_{rr} - \frac{\breve{H}_{rz}^2}{\breve{H}_{zz}}, H_{nn} = \frac{-1}{\breve{H}_{zz}}, H_{rn} = \frac{-\breve{H}_{rz}}{\breve{H}_{zz}}. \quad (4.1)$$

Describing equation (3.5.2) s.t (3.5.2a) in terms of (4.1)

$$H_t + N(t) r_R z - \left[r_R \left(\frac{\mu - r_R}{\pi^2\theta^2} + \frac{\mu_S - r_R}{\delta_S^2} \right) - \frac{\mu(\mu - r_R)}{\pi^2\theta^2} - \frac{\mu_S(\mu_S - r_R)}{\delta_S^2} + \frac{\mu - r}{2\pi^2\theta^2} + \frac{(\mu_S - r_R)}{\delta_S^2} \right] z^2 \breve{H}_{zz} = 0 \quad (4.2)$$

SUBJECT TO
$$\begin{aligned} & u_B(t) > 0 \\ & u_s(t) > 0 \\ & (1 - u_B - u_s) > 0 \\ & N(t) = I(t) W(t) > 0 \\ & r_R(cons \tan t) > 0 \end{aligned}$$
(4.2A)

Differentiating $\breve{H}$, with respect to z as in Chubing Zhang [5] and combining with $n = \varphi = -\breve{H}_z$ into (4.1) s.t (4.1a)

$$\varphi_t + r_R(\varphi + \varphi z) + \left[r_R \left(\frac{\mu - r_R}{\pi^2\theta^2} + \frac{\mu_S - r_R}{\delta_S^2} \right) - \frac{\mu(\mu - r_R)}{\pi^2\theta^2} - \frac{\mu_S(\mu_S - r_R)}{\delta_S^2} + \frac{\mu - r}{2\pi^2\theta^2} + \frac{(\mu_S - r_R)}{\delta_S^2} \right] (z^2\varphi_{zz} + 2z\varphi_z) = 0 \quad (4.\ 3)$$

Subject to
$$\begin{aligned} & u_B(t) > 0 \\ & u_s(t) > 0 \\ & (1 - u_B - u_s) > 0 \\ & N(t) = I(t) W(t) > 0 \\ & r_R(cons \tan t) > 0 \end{aligned}$$
(4.3a)

where the associated feed-back formula (i.e., the optimal Degenerate wealth generation strategies) are given by

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$$u_B^* = \frac{-z\varphi_z\left(\mu - r_R\right)}{\pi^2\theta^2\varphi} \tag{4.4}$$

$$u_S^* = \frac{-z\varphi_z\left(\mu_S - r_R\right)}{\delta_S^2\varphi} \tag{4.5}$$

Next is to solve the linear PDE, (4.3) s.t (4.3a), for φ, and substitute the solutions into (4.4) and (4,5), so as to obtain the optimal control strategies.

UTILITY FUNCTION TEST FOR DEGENERATE-PENSION WEALTH GENERATION STRATEGIES

To ascertain the PPM's level of satisfaction due to investment of the Degenerate-Pension wealth, we provide the explicit solution for the chosen utility. The change of variable technic was used to obtain the associated ODEs

Optimal Degenerate-Pension wealth generation strategies, with CRRA Utility Choice

We assumed that the PPM is a moderately risk-averse person, hence takes up power utility.

$$U\left(n\right) = \frac{n^\beta}{\beta}, \beta < 1, \beta \neq 0 \tag{5.1.1}$$

Following the approach in Chubing Zhang et al [5], we combine equations (4.2.8) and (5.1.1) to have

$$\varphi\left(T, r_R, z\right) = z^{\frac{1}{\beta-1}}, \beta < 1, \beta \neq 0$$
(5.1.1a)

Assuming a solution structure to equation (4.3) s.t (4.3a), e have

$$\varphi\left(t, r_R, z\right) = z^{\frac{1}{\beta-1}}\omega\left(t, r_R\right) + \phi(t) \tag{5.1.2}$$

such that

$$\phi(t) = 0, \omega(t, r_R) = 1 \tag{5.1.2a}$$

Obtaining $\varphi_t, \varphi_z \textit{ and } \varphi_{zz}$ in equation (4.3) s.t (4.3a)

$$\begin{aligned} \varphi_t &= z^{\frac{1}{\beta-1}}\omega_t + \phi'\left(t\right) \\ \varphi_z &= \frac{\omega z^{\frac{1}{\beta-1}}}{(\beta-1)z} \\ \varphi_{zz} &= \frac{\left(2-\beta\right)\omega z^{\frac{1}{\beta-1}}}{\left(\beta-1\right)^2 z^2} \end{aligned} \tag{5.1.3}$$

Substituting equations (5.1.2) and (5.1.3) into (4.3) s.t (4.3a), and simplifying, we obtain

$$\left\{z^{\frac{1}{\beta-1}}\omega_t + \phi'\left(t\right) + r_R\omega\left(t, r_R\right)z^{\frac{1}{\beta-1}} + r_R\phi\left(t\right) + \frac{r_R}{\beta-1}z^{\frac{1}{\beta-1}}L\left[\frac{2-\beta}{\left(\beta-1\right)^2}\omega + \frac{2\omega}{\beta-1}\right]z^{\frac{1}{\beta-1}}\right\} = 0 \, . \qquad \text{v(5.1.4)}$$

where,

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$$L = r_R\left(\frac{\mu - r_R}{\pi^2\theta^2} + \frac{\mu_S - r_R}{\pi^2\theta^2}\right) - \frac{\mu(\mu - r_R)}{\pi^2\theta^2} - \frac{\mu_S(\mu_S - r_R)}{\delta_S^2} + \frac{(\mu - r_R)}{2\pi^2\theta^2} + \frac{\mu_S - r_R}{\delta_S^2} \quad (5.1.4a)$$

Breaking up equation (5.1.4) due to its dependency on $z^{\frac{1}{\beta-1}}$

$$\left\{\omega_t + r_R\omega(t, r_R) + r_R\frac{1}{\beta - 1} + L\left[\frac{(2-\beta)\omega}{(\beta-1)^2} + \frac{2}{\beta-1}\omega\right]\right\} = 0 \quad (5.1.4b)$$

$$\phi'(t) + r_R\phi(t) = 0 \quad (5.1.4c)$$

Now, taking into the boundary condition (5.1.2a) and equations (5.1.4b) and (5.1.4c), we obtain

$$\omega(t) = e^{r_R(T-t)}\left[1 + \frac{m}{r_R}\right] - \frac{m}{r_R} \quad (5.1.4b(i))$$

$$\phi(t) = 0 \quad (5.1.4c(i))$$

Theorem 5.1

Suppose, equations (5.1.4b(i)), (5.1.4c(i)), and (5.1.2) hold. Then, the optimal Degenerate-Pension wealth invested in both mutual fund and stock are given by

$$u_B^* = \frac{-(\mu - r_R)\left[r_R z^{\frac{1}{\beta-1}} e^{r_R(\beta-1)}(r_R + m) - r_R(\beta-1)mz\right]}{\pi^2\theta^2\left(z^{\frac{1}{\beta-1}}\omega(t, r_R) r_R^2(\beta - t)\right)} \quad (5.1.5)$$

and

$$u_B^* = \frac{-(\mu_s - r_R)\left[r_R z^{\frac{1}{\beta-1}} e^{r_R(\beta-1)}(r_R + m) - r_R(\beta-1)mz\right]}{\delta_s^2\left(z^{\frac{1}{\beta-1}}\omega(t, r_R) r_R^2(\beta - t)\right)} \quad (5.1.6)$$

Proof

Putting (5.1.4b(i)), (5.1.4c(i)), and (5.1.2), the expected utility function becomes

$$\varphi(t, r_R, z) = z^{\frac{1}{\beta-1}} e^{r_R(T-t)}\left[1 + \frac{m}{r_R}\right] + \frac{m}{r_R} \quad (5.1.7)$$

Combining equations (5.1.4b(i)) and (5.1.3), we have

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$$f_z = \frac{e^{r_R(T-t)}\left[1+\frac{m}{r_R}\right]-\frac{m}{r_R}z^{\frac{1}{\beta-1}}}{(\beta-1)z} \quad (5.1.8)$$

Putting equations (5.1.8), and (5.1.7) into (4.4) and (4.5), we have

$$u_B^* = \frac{-(\mu-r_R)\left[r_R z^{\frac{1}{\beta-1}} e^{r_R(\beta-1)}(r_R+m)-r_R(\beta-1)mz\right]}{\pi^2\theta^2\left(z^{\frac{1}{\beta-1}}\omega(t,r_R)r_R^2(\beta-t)\right)} \quad (5.1.9)$$

and

$$u_B^* = \frac{-(\mu_s-r_R)\left[r_R z^{\frac{1}{\beta-1}} e^{r_R(\beta-1)}(r_R+m)-r_R(\beta-1)mz\right]}{\delta_s^2\left(z^{\frac{1}{\beta-1}}\omega(t,r_R)r_R^2(\beta-t)\right)}$$

(5.1.1.10), as required

FINDINGS

In equation (5.1.4c(i)), it was observed that the annuity term, due to the chosen utility vanished. This means that at a point in future, the inflation will eat up the little returns (annuity) the investor enjoys, to the extent that he will abruptly collapse his/her investment

CONCLUSION

Optimal Degenerate-Pension Wealth generation strategies for stock and mutual benefit account have been formulated. It was discovered herein that inflation has adverse impact on Pension wealth, hence the need to diversify the investment into many other assets becomes very necessary.

RECOMMENDATION

In order to hedge the investment against such sudden collapse, we diversify.

Further Work

There is need to simulate the real impact of the inflation on investments into stock and mutual benefit account

There is also need to simulate the effect of interest rate volatility on investments into stock and mutual benefit account

More so, there is need to simulate the effect of constant interest rate on investments into stock and mutual benefit account.

There is also need to repeat the whole process, using CARA utility function, and compare its results with CRRA utility function

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