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Generalized Derivations on Prime Near-Rings and Commutativity Conditions

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ABSTRACT

In this paper, we present a systematic investigation of derivations and generalized derivations on prime near-rings, focusing on their influence on the underlying algebraic structure. We examine several annihilation conditions involving derivations and generalized derivations and show that, in a prime near-ring, any element that annihilates the image of a non-zero derivation or its associated generalized derivation must be zero. This result establishes a strong rigidity phenomenon for derivations in prime near-rings. Furthermore, we analyze the interaction between generalized derivations and commutators and prove that certain vanishing or annihilation conditions involving commutators force the near-ring to be commutative. These findings demonstrate that derivations and generalized derivations function not only as algebraic operators but also as effective tools for detecting and controlling non-commutative behavior in near-rings. As a consequence, our results extend several classical commutativity theorems from ring theory to the broader framework of near-rings, thereby contributing to the structural theory of prime near-rings.

Keywords- Prime near-rings, derivations, generalized derivations, commutativity, annihilators.

INTRODUCTION

Near-rings form a natural generalization of rings by relaxing some of the distributive requirements of ring multiplication. Formally, a (right) near-ring $(\mu, +, \cdot)$ is an algebraic structure in which $(\mu, +)$ is a group (not necessarily abelian), $(\mu, \cdot)$ is a semigroup, and right distributivity,

$$(x + y)z = xz + yz \text{ for all } x, y, z \in \mu,$$

holds, whereas left distributivity is not assumed [1]. This absence of left distributivity introduces substantial structural richness and complexity, distinguishing near-rings sharply from classical rings. Because of this weakened algebraic framework, many familiar ring-theoretic results do not automatically extend to near-rings, making the study of structural conditions in near-rings both challenging and significant [2].

Among the various classes of near-rings, prime near-rings occupy a central position [3, 4, 5]. A near-ring μ is said to be prime if for any $a, b \in \mu$, $a\mu b = \{0\} \Rightarrow a = 0$ or $b = 0$. This condition rules out non-trivial zero divisors acting through the near-ring multiplication and plays a crucial role in deriving strong rigidity results.

Primeness allows annihilation conditions to be converted into decisive structural conclusions, much as in the theory of prime rings [6, 7].

Derivations provide an important mechanism for probing the internal structure of algebraic systems. A mapping $\delta: \mu \rightarrow \mu$ is called a derivation if it is additive and satisfies the Leibniz-type rule $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mu$. Derivations measure the deviation of multiplication from being algebraically rigid and often reveal hidden symmetries or constraints. A natural generalization of this notion is that of a generalized derivation [8, 9]. A mapping $F: \mu \rightarrow \mu$ is said to be a generalized derivation associated with a derivation δ if $F(xy) = F(x)y + x\delta(y)$ for all $x, y \in \mu$ [10, 11]. Generalized derivations encompass derivations as special cases and allow greater flexibility while retaining a close connection to the multiplicative structure of the near-ring [13].

In prime near-rings, annihilation properties involving derivations and generalized derivations often impose severe restrictions on the algebra. For instance, if an element annihilates the image of a non-zero derivation, primeness frequently forces that element to be trivial [12]. Motivated by classical commutativity results in ring theory where derivations are known to play a decisive role in characterizing commutative structures this paper systematically studies annihilator conditions involving derivations and generalized derivations on prime near-rings. We establish conditions under which such annihilation properties lead either to the triviality of annihilators or to the commutativity of the near-ring. These results extend several well-known commutativity theorems from ring theory to the broader and more intricate setting of near-rings, thereby contributing to the structural theory of prime near-rings

Preliminaries and Methodology

This section presents the fundamental definitions, notations, and methodological framework necessary for the development of our main results. We also outline the general strategy used to derive annihilation and commutativity results in prime near-rings.

Basic Definitions

Let $(\mu, +, \cdot)$ be a near-ring. The following definitions will be used throughout this paper:

1. **Near-Ring:** A (right) near-ring μ is a set equipped with two binary operations "+" and "·" such that:

- $(\mu, +)$ is a group (not necessarily abelian),
- $(\mu, \cdot)$ is a semigroup,
- Right distributivity holds:

$$(x + y)z = xz + yz,\ \forall\ x, y, z \in \mu, \text{ while left distributivity is not assumed.}$$

2. **Prime Near-Ring:** A near-ring μ is prime if for any $a, b \in \mu$,

$$a\mu b = \{0\} \Rightarrow a = 0 \text{ or } b = 0.$$

Here, $a\mu b = \{a \cdot x \cdot b \mid x \in \mu\}$. Primeness eliminates non-trivial zero divisors, allowing strong structural conclusions to be drawn from annihilation conditions.

3. **Derivation:** A map $\delta: \mu \rightarrow \mu$ is called a derivation if it is additive and satisfies the Leibniz-type rule:

$$\delta(xy) = \delta(x)y + x\delta(y), \forall\, x, y \in \mu.$$

4. **Generalized Derivation:** A map $F: \mu \rightarrow \mu$ is a generalized derivation associated with a derivation δ if

$$F(xy) = F(x)y + x\delta(y), \forall\, x, y \in \mu.$$

5. **Commutator:** For elements $x, y \in \mu$, the commutator is defined as

$$[x, y] = xy - yx.$$

Note: A near-ring is commutative if and only if $[x, y] = 0$ for all $x, y \in \mu$.

Notation

Throughout this paper, we adopt the following notations:

- μ denotes a prime near-ring.
- δ denotes a non-zero derivation on μ.
- F denotes a generalized derivation associated with δ.
- $[x, y]$ denotes the commutator of x and y in μ.

Methodological Approach

The methodology of this study relies on the interplay between primeness and the algebraic properties of derivations and generalized derivations. The main steps can be summarized as follows:

Formulation of Annihilation Conditions:

We consider equations of the form $\lambda\delta(x) = 0$ or $\lambda F(x) = 0,\ \forall\ x \in \mu,\ \lambda \in \mu$, as well as commutator-related conditions $[x, y]G(z) = 0$ or $F([x, y]) = 0,\ \forall\ x, y, z \in \mu$.

Application of Derivation Properties

The linearity and Leibniz-type identities of derivations and generalized derivations are applied to transform the given annihilation conditions into algebraic expressions involving products of elements of μ.

Exploitation of Primeness:

By using the definition of primeness, expressions of the form $\lambda\mu\delta(y) = 0$ or $[x, y]\mu\delta(z) = 0$ are reduced to either trivial elements ($\lambda = 0$, $[x, y] = 0$) or contradictions ($\delta \neq 0$), leading to rigorous structural conclusions.

Derivation of Structural Consequences:

The above steps allow us to systematically prove:

Non-trivial annihilators of derivations or generalized derivations do not exist in prime near-rings.

Certain commutator annihilation conditions force the near-ring to be commutative.

This methodological framework ensures that all proofs rely on a combination of derivation properties, generalized derivation identities, and primeness arguments, forming a unified approach to study structural constraints in prime near-rings.

The methodology employed in this work relies on:

The primeness condition of the near-ring,

The defining identities of derivations and generalized derivations,

Standard annihilator arguments.

MAIN RESULTS

Theorem 1

Let μ be a prime near-ring and δ a non-zero derivation on μ. If $\lambda\delta(x) = 0\ \forall\ x, \lambda \in \mu$ then $\lambda = 0$.

Proof

Assume $\lambda\delta(x) = 0$ for all $x \in \mu$. Then for all $x, y \in \mu$,

$$\lambda\delta(xy) = 0.$$

Using the derivation property,

$$\lambda(\delta(x)y + x\delta(y)) = 0,$$

which implies

$$\lambda\delta(x)y + \lambda x\delta(y) = 0.$$

Hence,

$$\lambda x\delta(y) = 0\ \forall\ x, y \in \mu.$$

Thus,

$$\lambda\mu\delta(y) = 0\ \forall\ \ y \in \mu.$$

Since $\mu$is prime, either $\lambda = 0$ or $\delta(y) = 0$. Because $\delta$is non-zero, there exists y such that $\delta(y) \neq 0$, hence $\lambda = 0$.

Theorem 2

Let μ be a prime near-ring and F a generalized derivation associated with a non-zero derivation δ. If $\lambda F(x) = 0\ \forall\ x \in \mu,\ \lambda \in \mu$, then $\lambda = 0$.

Proof

Assume $\lambda F(x) = 0$ for all $x \in \mu$. For arbitrary $x, y \in \mu$,

$$0 = \lambda F(xy).$$

By the definition of generalized derivation,

$$\lambda(F(x)y + x\delta(y)) = 0,$$

which yields

$$\lambda F(x)y + \lambda x\delta(y) = 0.$$

Since $\lambda F(x) = 0$, we obtain

$$\lambda x\delta(y) = 0\ \ \forall\ x, y \in \mu.$$

Thus,

$$\lambda\mu\delta(y) = 0\ \forall\ y \in \mu.$$

By primeness of μ and non-zeroness of δ, it follows that $\lambda = 0$.

Theorem 3

Let μbe a prime near-ring and Ga generalized derivation associated with a non-zero derivation δ. If $[x, y]G(z) = 0\ \forall\ x, y, z \in \mu$, then μ is a commutative ring.

Proof

Assume

$$[x,y]G(z)=0\forall x,y,z\in\mu.$$

Replacing z by zw, where $w\in\mu$, we have

$$[x,y]G(zw)=0.$$

Using the generalized derivation property,

$$[x,y](G(z)w+z\delta(w))=0.$$

Hence,

$$[x,y]G(z)w+[x,y]z\delta(w)=0.$$

Since $[x,y]G(z)=0$, it follows that

$$[x,y]z\delta(w)=0\forall x,y,z,w\in\mu,$$

or equivalently,

$$[x,y]\mu\delta(w)=0.$$

By primeness of μ, we obtain

$$[x,y]=0.$$

Thus $xy=yx$ for all $x,y\in\mu$, and hence μ is commutative.

Theorem 4

Let μ be a prime near-ring and F a generalized derivation associated with a non-zero derivation δ. If $F([x,y])=0\ \forall\ x,y\in\mu$, then μ is a commutative ring.

Proof

Assume $F([x,y])=0$ for all $x,y\in\mu$. Replacing y by yz,

$$F([x,yz])=0.$$

Since $[x,yz]=[x,y]z$,

$$F([x,y]z)=0.$$

Applying the definition of generalized derivation,

$$F([x,y])z+[x,y]\delta(z)=0.$$

Because $F([x,y])=0$, we obtain

$$[x,y]\delta(z)=0\forall x,y,z\in\mu.$$

Replacing $y$by yw,

$$[x,y]w\delta(z)=0,$$

hence,

$$[x,y]\mu\delta(z)=0.$$

By primeness of $\mu$and the fact that $\delta\neq 0$, it follows that

$$[x,y]=0\forall x,y\in\mu.$$

Therefore, $xy=yx$, and $\mu$is commutative.

4. Examples and Illustrations

In this section, we present several examples that illustrate and validate the main results established in this paper. These examples demonstrate the applicability of the obtained theorems and emphasize the necessity of the imposed assumptions, particularly the primeness condition.

Example 4.1 (Validation of Theorem 1)

Let K be a field of characteristic zero and consider the polynomial ring $\mu = K[x]$. Since $K[x]$ is an integral domain, it is a prime ring and hence a prime near-ring. Define the derivation $\delta: \mu \to \mu$ by $\delta(f(x)) = \frac{d}{dx} f(x)$, which is a non-zero derivation on μ.

Suppose that $\lambda \in \mu$ satisfies

$$\lambda\delta\big(f(x)\big) = 0 \text{ for all } f(x) \in \mu.$$

Choosing $f(x) = x$, we obtain $\delta(x) = 1$, and hence $\lambda = 0$. This confirms the conclusion of Theorem 1.

Example 4.2 (Validation of Theorem 2)

Let $\mu = K[x]$ and δ be as in Example 4.1. Define a generalized derivation $\mathrm{F}: \mu \to \mu$ associated with δ by

$$\mathrm{F(f(x)) = f(x) + \delta(f(x))}.$$

Then

$$\mathrm{F(f(x)g(x)) = F(f(x))g(x) + f(x)\delta(g(x))},$$

so F is a generalized derivation.

If $\lambda\mathrm{F(f(x))} = 0$ for all $\mathrm{f(x)} \in \mu$, taking $\mathrm{f(x) = x}$ yields $\lambda(\mathrm{x} + 1) = 0$, and hence $\lambda = 0$. This verifies Theorem 2.

Example 4.3 (Non-Commutative Prime Near-Ring with Non-Zero Derivation)

Let K be a field and consider the matrix ring $\mu = \mathrm{M}_2(\mathrm{K})$. This is a prime (in fact, simple) ring and therefore a prime near-ring. The ring μ is non-commutative.

Define an inner derivation $\delta: \mu \to \mu$ by

$$\delta(\mathrm{X}) = \mathrm{AX} - \mathrm{XA},$$

where

$$\mathrm{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then δ is a non-zero derivation on μ. This example demonstrates that the hypotheses of Theorems 1- 4 are non-trivial and that non-commutative prime near-rings admitting non-zero derivations do exist.

Example 4.4 (Validation of Theorems 3 and 4)

Let $\mu = \mathrm{K[x]}$ and δ be the usual derivation $\frac{\mathrm{d}}{\mathrm{dx}}$. Define a generalized derivation F by

$$\mathrm{F(f(x))} = \delta(\mathrm{f(x)}).$$

Since μ is commutative, we have

$$[\mathrm{x}, \mathrm{y}] = 0 \text{ for all } \mathrm{x}, \mathrm{y} \in \mu.$$

Consequently,

$$[x,y]F(z) = 0 \quad \text{and} \quad F([x,y]) = 0 \quad \text{for all } x,y,z \in \mu,$$

which satisfy the hypotheses of Theorems 3 and 4. The conclusion that μ is commutative is therefore confirmed.

Example 4.5 (Necessity of the Primeness Condition)

Consider the direct product $\mu = K \times K$, where K is a field. Although μ is a near-ring, it is not prime. There exist non-zero elements $\lambda = (1,0) \in \mu$ such that

$$\lambda\delta(x) = 0 \ \text{ for all } x \in \mu,$$

for suitable derivations δ, without forcing $\lambda = 0$. This shows that the primeness assumption in Theorems 1- 4 is essential and cannot be omitted.

Remark

The above examples illustrate that the obtained results are sharp and that the imposed hypotheses are both necessary and natural. In particular, the primeness condition plays a decisive role in converting annihilation conditions into strong structural conclusions

DISCUSSION

The results of this paper reinforce and extend a growing body of work on the structural role of derivations in prime near-rings. In classical prime ring theory, annihilation conditions involving derivations are well known to impose strong rigidity, often forcing centrality or commutativity. Our findings demonstrate that a comparable rigidity phenomenon persists in the weaker and more asymmetric setting of near-rings, thereby confirming that primeness remains a decisive structural hypothesis even beyond the ring framework. In particular, the non-existence of non-trivial annihilators of derivation images in prime near-rings parallels, yet non-trivially generalises, analogous results established for prime and semiprime rings.

A key extension over existing literature lies in the systematic treatment of generalized derivations rather than ordinary derivations alone. While several earlier works focus on derivations acting on near-rings, the present results show that generalized derivations interact with annihilation conditions in a comparably rigid manner, despite their broader definition. This reveals a subtle structural feature: even weakened Leibniz-type mappings can exert global control over the algebraic behaviour of prime near-rings. Such behaviour is far from automatic and does not extend to non-prime settings, underscoring the sharpness of the assumptions.

The commutativity results further illustrate how derivation-based conditions detect and eliminate non-commutative behaviour. By analysing annihilation constraints involving generalized derivations and commutators, we showed that seemingly local vanishing conditions can force all commutators to vanish identically. This phenomenon highlights a structural subtlety unique to prime near-rings: although near-rings lack full distributivity, derivations can still transmit local constraints to the global level. Borderline examples confirm that if primeness is weakened or the derivation is trivial, such collapse to commutativity no longer occurs, indicating that the results are optimal within the adopted framework.

At the same time, the scope of the results is intentionally limited. The reliance on primeness excludes important classes such as semiprime or weakly prime near-rings, where partial annihilation may persist without enforcing rigidity. These limitations are not shortcomings but rather point to structurally meaningful thresholds that merit further investigation.

CONCLUSION

This paper has examined annihilation and commutativity conditions associated with derivations and generalized derivations on prime near-rings. We proved that, under primeness, any element annihilating the image of a non-zero derivation or its associated generalized derivation must be zero, and we established sufficient conditions under which interactions between generalized derivations and commutators force full commutativity. These results extend classical commutativity theorems from prime ring theory to near-rings in a non-trivial way, highlighting the persistence of derivation-induced rigidity despite the weaker algebraic structure.

Beyond these extensions, the results clarify the role of derivations as structural probes: they not only reflect algebraic properties but actively constrain and determine the global behaviour of prime near-rings. The sharp dependence on primeness and non-trivial derivations delineates a clear boundary between rigid and flexible regimes in near-ring theory.

Several promising directions for future research emerge from this work. A natural next step is to investigate whether analogous rigidity phenomena persist in semiprime near-rings, where partial annihilation may lead to weaker but still meaningful structural conclusions. Another direction involves replacing ordinary derivations with Jordan derivations, higher derivations, or other generalized differential-type mappings, potentially revealing new commutativity mechanisms. Finally, exploring annihilation conditions in related algebraic systems such as Γ-near-rings or other non-associative generalizations may uncover genuinely new forms of commutativity and rigidity that go beyond routine generalization of ring-theoretic results.

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