Numerical Solution to the System of Integro-Differential Equations Using Collocation Approximation

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ABSTRACT

The system of integro-differential equations (IDEs) has been utilized to model a wide range of problems in finance, control systems, viscoelasticity, engineering, and the wide-ranging applications of these equations in various fields of science. In this paper, we present efficient solvers for solving a system of high-order linear Volterra–Fredholm integro-differential equations (VFIDEs) using numerical techniques. The system of integro-differential equations is reduced to equations into a system of linear algebraic equations, and matrix inversion is employed to solve the algebraic equations. Only a small number of S-polynomials are needed to obtain a satisfactory result. The method's error analysis is presented. Several examples are provided to demonstrate the use of the collocation approach. The numerical simulation demonstrates the dependability and efficiency of the collocation method. The proposed method is highly effective, straightforward, and well-suited for solving systems of Volterra integro-differential equations.

Keywords: S-polynomials, Integro-differential equations, Operational matrix, Collocation method.

INTRODUCTION

Over the past few decades, there has been a growing prevalence of systems of integro-differential equations in mathematical models [1][2]. Numerous researchers have dedicated their efforts to developing analytical methods, and numerical methods to solve these equations, due to the wide-ranging applications of these equations in various fields of science. The system of Integro-differential equations was used to model the competition between tumor cells and the immune system [3]. Involving fluid waves with an oceanographic application was reduced to a system of integro-differential equations [4]. A system of specific, inhomogeneous Integro-differential equations was studied to observe the phenomenon of the noise term [5]. The mathematical representation of numerous natural and engineering processes depends extensively on the system of Volterra Integro-differential equations [6]. Some of the applications of fundamental issues in statistical and continuum mechanics are investigated [7]. The use of the system of Volterra integro-differential equations in the modelling of engineering and physical phenomena to solve real-world problems can be found in [8] [9] [10]. A number of researchers have investigated differential equations in their research, particularly in [11] [12].

The study of IDEs is mostly driven by practical applications. Some scientific and engineering phenomena, like the theory of dynamical systems with automatic control [13], the kinetic theories of rarefied gases, plasma, radiation transfer, coagulation, the study of stochastic processes with jumps, and more specifically of Levy processes, which are expressed by IDEs [14], are modeled in a significant manner by the inclusion of Integro-differential equation. Numerous processes in the applied sciences, including physics, chemistry, economics, control theory, mechanics, biology, and engineering, are modelled by integro-differential equations [15][16]. arise in many physical phenomena such as wind ripples in the desert, Nanohydrodynamics, population growth models, glass-forming process, and oceanography [17][18][19]. The study of FIDEs has attracted attention in recent years. Several significant findings regarding the corresponding inequalities of Integro-differential equations have been shown [20]. The system of integro-differential equations has motivated extensive research in recent years. In conventional models, several types of Integro-differential equations have been developed, and there has been considerable interest in developing numerical approaches for their solutions [21] [12]. Various numerical methods for solving systems of integro-differential equations have been developed by many researchers. In the past decades, integro-differential equations have been solved using a variety of techniques, such as the Adomian decomposition method [22], the variational iteration method and homotopy perturbation method [23], wavelets [24] [25], operational Tau method [26], and others. Then several numerical methods have been used, such as the Chebyshev polynomial method [27], Galerkin methods with hybrid functions [28], the tau method [6], Runge–Kutta methods [29], the spline approximation method [30], the spectral method [31], the finite difference approximation method [32], etc.

Also, Several methods have been used to pursue the solution of Integro-differential equations, including the Adomian decomposition method developed [33] [34], the Hybrid linear multistep method [35] [36], the Chebyshev-Galerkin method [37], the Bernoulli matrix method [38], the Bernstein Polynomials Method, the Legendre collocation method [39] and fractional-order Genocchi deep neural networks [40] Polynomials perform a crucial role in numerous fields of mathematics. These polynomials have been employed in the solution of integral equations, differential equations, integro-differential equations, and systems of integro-differential equations, as well as in approximation theory; see e.g. [41][42][43][44][45]. There are various types of polynomials for solving various problems. Such as Hahn polynomials for non-singular coupled Burgers' equations [46], Müntz-Legendre polynomials for variational problems [2,28], and recently, Clique polynomials for time-fractional Klein-Gordon equations [47]. But there is no method in the literature for systems of integro-differential equations using the S-polynomials.

In this paper, a family of polynomials called S-polynomials, is introduced. We have explored the properties and uses of these polynomials and developed strategies for solving systems of integro-differential equations. because its implementation is straightforward. We approximate the solution system of the Integro-differential equation in this paper using s-polynomials as a basis in conjunction with the collocation approach to numerically solve the proposed problem. The operational matrix for a polynomial has been developed to cover the numerical solution of a system of Integro-differential equations. We derive the operational matrices for derivatives and integral terms using s-polynomials.

To obtain the desired results, the s-operational matrices of derivatives in integers are introduced and used in conjunction with the collocation method. By implementing s-polynomial approximations and the operational matrices of derivative and integral terms, the original problem has been reduced to a set of algebraic equations. The numerical solution of FIDE can be obtained by solving this system of algebraic equations at collocation points. to acquire a system of algebraic linear equations was collocated at appropriate points within the interval of investigation [0,1]. This paper is organized as follows: In Section 2, the preliminary. In Section 3, we introduce S-polynomials and their properties. In Section 4, the solution method is presented. In Section 5, convergence analysis.

In Section 6, numerical examples are used to illustrate the convergence, accuracy, and advantages of the proposed method, and compare it with others. Finally, in Section 7, the conclusion of the work is given.

Preliminaries

In this research, we consider a system of Integro-differential equations of the form:

$$\sum_{n=0}^{m}\sum_{j=1}^{N} P_{ij}^{n}(x)\, y_j^{(n)}(x) = f_j(x) + \int_{h(x)}^{g(x)} \sum_{j=1}^{N} k_{ij}(x,t) y_j(t)dt \qquad i = 1,2,3,\dots,N \qquad (1)$$

A system of integro-differential equations (IDEs) consists of equation in which the unknown function appears both within a derivative and under an integral sign.

Where $P_{ij}^{n}(x)$ are known real valued functions, $y_j^{(n)}(x)$ is the nth-derivative of $y(x)$ with order n, $y_j(x)$ is an unknown function, $k_{ij}(x,t)$ is a kernel function, and $f_j(x)$ is a linear or non-linear function.

Differential Equation

A differential equation is a mathematical equation that relates one or more unknown functions to their derivatives.

$$\sum_{n=0}^{m}\sum_{j=1}^{N} P_{ij}^{n}(x)\, y_j^{(n)}(x) = f_j(x) \qquad i = 1,2,3,\dots,N \qquad (2)$$

A differential equation is said to be linear if it is of first degree (degree one) and there is no product of the dependent variable and its derivative(x). Otherwise, it is called non-linear.

Integro-Differential Equation

An Integro-differential equation is a type of mathematical equation that combines both differential and integral operators. It involves functions that depend on both the values of its function and its derivative at a given point, as well as the integral function over a specified interval. An integro-differential equation is classified into two forms:

Volterra Integro-Differential Equation

where h (x) and g (x) are the limits of integration. Or if at least one limit is a variable, then it is said to be a Volterra integral equation, [5].

$$\sum_{n=0}^{m}\sum_{j=1}^{N} P_{ij}^{n}(x)\, y_j^{(n)}(x) + \int_{h(x)}^{g(x)} \sum_{j=1}^{N} k_{ij}(x,t) y_j(t) dt = f_j(x) \qquad i = 1,2,3,\dots,N \qquad (3)$$

Fredholm Integro-Differential Equation

An integro-differential equation is said to be a Fredholm equation if it is of the form where the limits are constants.

$$\sum_{n=0}^{m}\sum_{j=1}^{N} P_{ij}^{n}(x)\, y_j^{(n)}(x) + \int_{a}^{b} \sum_{j=1}^{N} k_{ij}(x,t) y_j(t) dt = f_j(x) \qquad i = 1,2,3,\dots,N \qquad (4)$$

S-Polynomials

Polynomials play a crucial role as a vital tool in numerical methods for solving practical problems. Due to the characteristics of the S-polynomials, these polynomials are of interest for finding approximate solutions to problems of integer order differential equations. It has an important feature: the S-polynomials are derivable.

The S-polynomials of degree n are defined by

$$S_n(x) = \left(x - \frac{1}{2}\right) S_{n-1}(x) - \frac{1}{n} S_{n-2}(x), \qquad for\ n \geq 2 \qquad (5)$$

$$S_1(x) = 1$$

$$S_2(x) = x - \frac{1}{2}$$

$$S_3(x) = (x)^2 - x - \frac{1}{12}$$

$$S_4(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{6}x + \frac{1}{6}$$

$$S_5(x) = x^4 - 2x^3 + \frac{43}{60}x^2 + \frac{17}{6}x - \frac{1}{15}$$

METHODOLOGY

In this section, we will utilize the S-polynomial for the system of Volterra-Fredholm integral equations. The operational matrix of integration is derived, the product operational matrix is introduced, and the operational matrix of derivatives for s-polynomials is shown.

$$y(x) = \sum_{j=0}^{N} C_j S_j(x)$$

$$=c^T S(x)$$

$$=[c_1, c_2, c_3, \dots, c_N]^T [S_1(x), S_2(x), S_3(x), \dots, S_N(x)]^T$$

Vector coefficient $[c_1, c_2, c_3, \dots, c_N]^T$ and vector polynomial $[S_1(x), S_2(x), S_3(x), \dots, S_N(x)]^T$

where c_i, i = 0, 1, 2, . . .N are the series unknown coefficients.

Differentiation P-Operational Matrix N × N

Operational matrices of derivatives for S-polynomials. This section examines the operational matrices of S-polynomials for integer-orders of derivatives. A function y(x), expressed in terms of S-polynomials, can be approximated as follows:

$$\begin{aligned} y(x) &= \sum_{j=0}^{N} c_j S_j(x) \\ y'(x) &\approx \sum_{j=0}^{N} D\left(c_j S_j(x)\right) \\ &= \sum_{j=0}^{N} c_j D S_j(x)\,, \end{aligned}$$

$$\begin{aligned} DS_j(x) &= \sum_{i=0}^{N} P_{i,j} S_i(x) \\ y'(x) &= \sum_{j=0}^{N} c_j \sum_{i=0}^{N} P_{i,j} S_i(x) \\ &= \sum_{i=0}^{N} \sum_{j=0}^{N} P_{i,j} c_j S_i(x) \end{aligned}$$

$$= \sum_{i=0}^{N} CP_i S_i(x)$$

then, for the nth derivative of $y^{(n)}(x)$

$$\begin{aligned}
y^{(n)}(x) &= C_k S_k'(x) \\
&= \mathrm{C\ D\ S(x)} \\
&= C\ D \begin{vmatrix} S_{11(x)} & S_{12(x)} \cdots & S_{1N(x)} \\ S_{21(x)} & S_{22(x)} \cdots & S_{2N(x)} \\ \vdots & \vdots & \vdots \\ S_{N1(x)} & S_{N2(x)} \cdots & S_{NN(x)} \end{vmatrix} \\
&= C \begin{bmatrix} D(S_{11(x)}) & D(S_{12(x)}) & \cdots & D(S_{1N(x)}) \\ D(S_{21(x)}) & D(S_{22(x)}) & \cdots & D(S_{2N(x)}) \\ \vdots & \vdots & & \vdots \\ D(S_{N1(x)}) & D(S_{N2(x)}) & \cdots & D(S_{NN(x)}) \end{bmatrix} \\
&= \mathrm{C\ P} \begin{bmatrix} S_{1(x)} \\ S_{2(x)} \\ \vdots \\ S_{N(x)} \end{bmatrix} \\
&= c^T \mathrm{P\ S(x)}
\end{aligned}$$

Integration K- operational matrix N × N

Operational matrices of integrals for S-polynomials. This section examines the operational matrices of S-polynomials. A function y(x), expressed in terms of S-polynomials, can be approximated as follows:

$$\begin{aligned}
y(x) &= \sum_{j=0}^{N} c_j S_j(x) \\
\int_0^X y(x) &\approx \int_0^X \sum_{j=0}^{N} c_j S_j(x) \\
&= \sum_{j=0}^{N} c_j \int_0^x S_j(x)\,,
\end{aligned}$$

$$\begin{aligned}
\int_0^x S_j(x) &= \sum_{i=0}^{N} k_{i,j} S_i(x) \\
\int_0^X y(x) &= \sum_{j=0}^{N} c_j \sum_{i=0}^{N} k_{i,j} S_i(x) \\
&= \sum_{i=0}^{N} \sum_{j=0}^{N} k_{i,j} c_j S_i(x)
\end{aligned}$$

$$= \sum_{i=0}^{N} CP_i S_i(x)$$

$$\int_0^X y(x) = \sum_{i=0}^{N} CP_i S_i(x)$$

Implementation of the proposed scheme for the specified problem.

$$\sum_{n=0}^{m}\sum_{j=1}^{N} P_{ij}^{n}(x)\, y_j^{(n)}(x) = f_j(x) + \int_{h(x)}^{g(x)} \sum_{j=1}^{N} k_{ij}(x,t) y_j(t)dt \qquad i = 1,2,3,\dots,N \tag{6}$$

Algorithm

Input

1. N, M, I, j
2. $f_j(x)$, $k_{ij}(x,t)$
3. Initial condition $y_j(t)$
4. Initial condition $y_j^{(n)}(x)$
5. Define y(x) $= \sum_{j=0}^{N} c_j S_j(x)$
6. Substitute y(x) in each equation
7. Select appropriate collocation points
8. Solving the algebraic system to get c_{ij}
9. Substitute c_{ij} into y(x)
10. Input y(x) Exact
11. Plot y(x) the exact and y(x) the approximation
12. Define the error as |y(x) exact − y(x) approximation |
13. Plot the error function

Convergence Analysis

$$\sum_{n=0}^{m} P_i\,(x) y^{(n)}(x) = f(x) + \int_0^x k(x,t)y(t)dt, \qquad i = 1,2,3,\dots,N$$

Assume $y_N(x)$ is the approximate solution and $y(x)$ is the exact solution then

$f(x)$ and $k(x,t)$ are continuous function if and only if

$$\frac{\left\|\sum_{i=0}^{N} P_i\,(x)(y_N(x) - y(x))\right\|_\infty}{\|y_N(x) - y(x)\|_\infty} \le sup \int_0^x |k(x,t)|dt$$

Proof: let the exact solution

$$\sum_{i=0}^{N} P_i\,(x)y(x) = f(x) + \int_0^x k(x,t)y(t)dt \tag{7}$$

Substituting the approximate solution $y_N(x)$ into eq.7 then

$$\sum_{i=0}^{N} P_i\,(x)y_N(x) = f(x) + \int_0^x k(x,t)y_N(t)dt \tag{8}$$

Subtracting (7) from (8) gives

$$\sum_{i=0}^{N} P_i\,(x)(y_N(x) - y(x)) = \int_0^x k(x,t)(y_N(t) - y(t))dt$$

Absolute value

$$\left|\sum_{i=0}^{N} P_i\,(x)(y_N(x) - y(x))\right| \le sup \int_0^x |k(x,t)||y_N(t) - y(t)|dt$$

$$\left\|\sum_{i=0}^{N} P_i\,(x)(y_N(x) - y(x))\right\|_\infty \le sup \int_0^x |k(x,t)|\|y_N(t) - y(t)\|_\infty dt$$

$$\frac{\left\|\sum_{i=0}^{N} P_i\,(x)(y_N(x) - y(x))\right\|_\infty}{\|y_N - y\|_\infty} \le sup \int_0^x |k(x,t)|dt$$

$$\frac{\left\|\sum_{i=0}^{N} P_i\,(x)(E(x))\right\|_\infty}{\|E\|_\infty} \le sup \int_0^x |k(x,t)|dt \qquad (9)$$

Illustrative Examples

The focus of this section is to demonstrate the method's effectiveness, efficiency, and simplicity. We will test the applicability of the provided approach in this research for solving higher-order Integro-differential equations via several different examples, using a Python program. We state the numerical results in the tables and figures below.

Example 1.

$$y'(x) + \frac{3x^4}{600} - \frac{\sqrt{4}}{200} = \int_0^x t^2 y(t)dt$$

Table 1: Numerical results and absolute errors for Example 1

x_i	Rung-Kutta4	N=3		N=6	
		y	**Absolute errors**	**y**	**Absolute errors**
0	0	0	0	0	0
0.2	0.00200000	0.00199953	4.70E-07	0.00199979	2.10E-07
0.4	0.00400000	0.00399850	1.50E-06	0.00399974	2.60E-07
0.6	0.005997000	0.00599389	3.11E-06	0.00599673	2.70E-07
0.8	0.007989000	0.00798372	5.28E-06	0.00798879	2.10E-07
1	0.009973000	0.00996487	8.13E-06	0.00997292	8.00E-08

Example 2.

$$y_1'(x) = 1 + \frac{3}{2}x^2 + \frac{1}{3}x^3 - x^5 - x + y_1(x) + y_2(x) - \int_0^x (y_1(t) + y_2(t))dt$$

$$y_2'(x) = 2x - \frac{1}{3}x^3 - \frac{1}{4}x^4 + \int_0^x t(y_1(t) + y_2(t))dt$$

Table 2: Comparison of the numerical results and the Rung-Kutta4 results for Example 2.

x_i	Rung-Kutta4		N=3		N=6	
	y_1	y_2	y_1	y_2	y_1	y_2
0	0	0	0	0	0	0
0.2	0.20557700	0.04000200	0.16648746	0.03834123	0.20574291	0.03999378
0.4	0.44613400	0.16009800	0.41458701	0.15941427	0.44612561	0.16000993
0.6	0.75681400	0.36113200	0.74429864	0.36321913	0.75687791	0.36112916
0.8	1.15971900	0.64637800	1.15562351	0.64975579	1.16021305	0.64635661
1	1.63964600	1.02395400	1.64855814	1.01902476	1.64027475	1.02390163

Table 3: Absolute error values for Example 2.

x_i	Absolute errors N=3		Absolute errors N=6	
	y_1	y_2	y_1	y_2
0	0	0	0	0
0.2	3.91E-02	1.66E-03	1.66E-04	8.22E-06
0.4	3.15E-02	6.84E-04	8.39E-06	8.81E-05
0.6	1.25E-02	2.09E-03	6.39E-05	2.84E-06
0.8	4.10E-03	3.38E-03	4.94E-04	2.14E-05
1	8.91E-03	4.93E-03	6.29E-04	5.24E-05

Example 3. A system of third – order linear Volterra–Fredholm integro-differential equations (VFIDEs)

$$y_1'''(x) + y_2''(x) - y_3(x) = f_1(x) + \int_0^x t\big(y_2(t)dt - y_2(t)\big)dt - \int_0^1 x^2 t y(t)dt - \int_0^1 y_3(t)dt$$

$$\cos(x)\, y_3'''(x) - 2y_3'(x) - x - y_1(x) = f_2(x) - \int_0^x (1-t)y_1(t)dt - 2\int_0^x \sin(t) y_2(t)dt - 2\int_0^1 \sin(t) y_3(t)dt$$

$$y_2'(x) - y_1''(x) - y_1(x) = f_3(x) - \int_0^x y_2(t)dt - \int_0^x \tan(t)\, y_3(t)dt + \int_0^1 e^t y_1(t)dt$$

$$f_1(x) = x^2(1 - 2\cos(1) + \sin(1)$$

$$f_2(x) = x + sin^2(1) + 2xsin(x)$$

$$f_3(x) = 1 - ecos(1) + \cos(x)$$

Table 4: Absolute error values for Example 2.

x_i	Absolute errors for y_1		Absolute errors for y_2		Absolute errors for y_3	
	$N = 3$	$N = 6$	$N = 3$	$N = 6$	$N = 3$	$N = 6$
0	4.85E-02	9.23E-04	1.00E-02	5.62E-03	1.75E-01	4.81E-05
0.2	3.88E-02	1.27E-02	1.16E-03	4.40E-03	1.03E-01	1.75E-03
0.4	2.16E-02	2.16E-02	1.63E-02	7.84E-03	5.28E-02	2.67E-03
0.6	7.87E-03	2.39E-02	3.63E-02	4.73E-03	1.91E-02	2.72E-03
0.8	5.57E-02	1.28E-02	4.26E-02	1.54E-03	8.88E-04	1.73E-03
1	1.28E-01	2.13E-02	1.75E-02	4.78E-03	9.13E-03	5.97E-04

CONCLUSION

S-polynomials were presented for solving systems of high-order linear VFIDEs. It was applied and studied using numerical examples of various orders, and the numerical technique was employed in numerical experiments. To demonstrate the proposed method's efficiency and applicability, the results of Example 1 are shown as the values of N increase, indicating that the approximate solutions converge to the exact solution, as shown in Table 1. In examples 2 and 3, the approximate solution at N=3 is obtained, and by solving for N=6, we obtain Table 2, which shows the results at x=0 compared with the exact solution. Examples 2 and 3 shows that, as the worth of N rises, the error decreases, as shown in Tables 3 and 4. Finally, the agreement in numerical solutions is supported by the numerical results, which demonstrate the validity and accuracy of the method, convenient for solving systems of high-order VFIDEs. All computations are done with the aid of Python.

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