Fixed Point Theorems in Metric Spaces Induced by Coprime Relations

Dr. Raja Ram Singh

Assistant Professor, Department of Mathematics, S. D. College Kaler, Arwal, Magadh University, Bodh-Gaya, Bihar

ORCID-0009-0004-4590-8191

DOI: https://doi.org/10.51244/IJRSI.2026.1303000147

Received: 08 December 2025; Accepted: 18 December 2025; Published: 09 April 2026

ABSTRACT

In this recent study, we investigated a novel fixed point in a special metric space where the distance between two natural numbers is determined by their coprime relations. We define a metric related to the Greatest Common Divisor (GCD) and explore fixed-point results for standard coprime controlled contraction mappings and their generalizations as well as fixed point results using additional mathematical structures. With the support of artificial intelligence and we also introduced Banach, Cric, and Kannan applications with examples to illustrate this theory.

Keywords: Fixed point, metric space, contraction mapping, number theory.

INTRODUCTION

Fixed-point theory remains a central topic in mathematical analysis, with extensive applications in nonlinear analysis, dynamic systems, game theory, and mathematical modelling. A fixed point of a function is a value that is mapped to itself using the function. Banach's contraction principle is one of the most well-known results in this domain, ensuring the existence and uniqueness of fixed points under strict contractive conditions in complete metric spaces. Recently, research has focused on refining and generalising fixed-point results using additional mathematical structures, such as order theory, graph theory, and fuzzy logic. This study introduces a novel approach by integrating number theory, specifically the concept of coprime numbers, into fixed point theory. The main idea is to use the arithmetic structure of coprimality to define new types of contractive mappings and iteration schemes in metric spaces. We investigate the implications of these coprime-induced mappings, provide sufficient conditions for the uniqueness of fixed points and explore the interaction between the metric space topology and the arithmetic properties of natural numbers. This interdisciplinary approach opens new avenues for mathematical analyses and applications. Fixed point theorems

were first developed by Banach [1] in 1922. A fixed point theorem for mappings that did not increase distances was proposed by Kirk [2] in 1965.Commuting mappings and fixed points by Jungck [3] in 1976. Fixed point theory for Lipschitzian-type mapping Agarwal, O'Regan and Sahu [4] in 2009. An introduction to the theory of numbers by Hardy and Wright [5] in 2008. Introduction to Metric Spaces and fixed point theor by Khamsi, and Kirk [6] in 2001.Fixed point theorems via altering distance functions by Gopal and Rhoades [7] in 2011. Coprime structures in metric dynamics by Smith [8] in 2014. Generalized contractions and fixed point theorems by Altun and Sadarangani [9] in 2012.An Iterative approximation of fixed points by Berinde [10] in 2007. Fixed point theorems for mappings satisfying inwardness conditions by Caristi [11] in 1976. Nonexpansive projections on subsets of Banach spaces by Bruck [12] in 1973. Fixed point theorems in partially ordered metric spaces using coprime functions by Zafar and Rao in 2018. A generalization of the contraction principle in metric spaces by Kosmol, [13] in 1987. A fixed point theorem in complete metric spaces by Gregus [14] in 1980. Common fixed points for weakly compatible maps in fuzzy metric spaces by Abkar and Eslamian [15] in 2010. Fixed point theory in metric spaces and application studies have been introduced [16, 17, 18]. Our main purpose is to demonstrate that the contribution lies in defining coprime

ISSN No. 2321-2705 | DOI: 10.51244/IJRSI |Volume XIII Issue III March 2026

controlled contraction functions and exploring their fixed point properties through rigorous proofs. This study introduces new types of mappings and provides sufficient conditions for the uniqueness of the fixed- points.

PRELIMINARIES

Coprime Numbers : Let a and b are two coprime integers if GCD (a, b) = 1.

Metric Space: A metric space is a pair (X, d), where d: X×X→ R, then the following conditions are satisfied:

- d(x, y) = 0, iff x = y,
- d(x, y) = d(y, x) (symmetry)
- d(x, z) ≤ d(x, y) + d(y, z) (triangular inequality)

Fixed Point: Let F: X→X. A point x ∈ X is called a fixed point of F if F(x) = x.

Metric Based on Coprime Numbers:

We define a metric d on the set of natural numbers N as:

- d(a, b) = 0 if, a = b
- GCD (a, b) =1 if, a ≠ b
- d(a, b) = 2 otherwise

Explanation:

- Distance 0 if number are the same.
- Distance 1 if different but co-prime.
- Distance 2 if not coprime.

Metric properties:

- Non-negative: obvious
- Identity of indiscernibies : d(a, b)=0
- symmetry: d(a, b) = d(b, a) because GCD (a, b)= GCD (b, a)
- Triangle enequality: If a = b or b= c or a= c it is easy,
- If a, b, c are distinct:

If GCD (a, b) = 1 and GCD (b, c) = 1,

then d(a, b) = 1 ,d(b, c)=1, so, d(a, c) ≤ 2

In the worst case, d(a, c) = 2 ≤ 1+1 = 2

Let take some integers:

a = 4, b = 9 ⇒ GCD(4, 9) = 1 ⇒ d(4, 9) = 1

a = 6, b = 9 ⇒ GCD(6, 9) = 3 ⇒ d(6, 9) = 2

This metric is discrete and reflects a number theoretic distance.

Thus, (N, d) is a metric space.

MAIN RESULTS

3.1 Theorem: (Banach fixed point in coprime metric space)

Let (N, d) be the metric space as described above. Suppose that F: N→N satisfies:

d(F(x), F(y)) ≤ k d(x, y) for all x, y ∈ N, where 0 ≤ k < 1.

ISSN No. 2321-2705 | DOI: 10.51244/IJRSI |Volume XIII Issue III March 2026

Then F has a unique fixed point.

Proof: Because N is discrete with this metric, each Cauchy sequence is eventually constant. Thus, any contraction mapping F must eventually be fixed at a point. Specifically, starting from any $x_0 \in N$, we define: $x_{n+1} = F(x_n)$. Then,

$d(x_{n+1}, x_n) = d(F(x_n), F(x_{n-1})) \leq kd(x_n, x_{n-1})$

Thus, $d(x_n, x_{n-1})$ tends to 0, implying x_n stabilizes.

Hence, there exists $x^* \in N$ such that $F(x^*) = x^*$.

Now we uniqueness:

If x^*, y^* are two fixed points. Then

$d(x^*, y^*) = d(F(x^*), F(y^*)) \leq kd(x^*, y^*)$

But, $0 \leq k < 1$, so $d(x^*, y^*) = 0$, implying $x^* = y^*$.

Hence, the proved.

Theorem: (Coprime contraction fixed point theorem)

Statement: Let (X, d) be a complete metric space and $F: X \rightarrow X$ a mapping satisfying

$d(Fx, Fy) \leq k \cdot \frac{d(x,y)}{(1+GCD(\phi(x),\phi(y))}$,$\forall$ x, y $\in$ X, Then F has a unique fixed point in X.

Proof : Given: $F: X \rightarrow X$ is a self-mapping on a complete metric space (X, d).

Assumption: $d(Fx, Fy) \leq k \cdot \frac{d(x,y)}{(1+GCD(\phi(x),\phi(y))}$

we have: $d(Fx, Fy) \leq \frac{K}{2} d(x, y)$.

Define a sequence:

An arbitrary point $x_0 \in X$ is choosen .We define a sequence $\{x_n\}$ by iteration: $x_{n+1} = F(x_n)$, for $n \geq 0$.

Estimate the distance between successive terms:

Using the contraction condition,

$d(x_{n+1}, x_n) = d(Fx_n, Fx_{n-1}) \leq \frac{K}{2} d(x_n, x_{n-1})$.

Then we obtain by induction methods $d(x_{n+1}, x_n) \leq (\frac{K}{2})^n d(x_1, x_0)$.

Now consider the total distance from x_n to x_m:

$d(x_n, x_m) \leq \sum_{i=m}^{n-1} d(x_{i+1}, x_i) \leq d(x_1, x_0) \sum_{i=m}^{n-1} (\frac{K}{2})^i$.

$d(x_n, x_m) \leq \sum_{i=m}^{n-1} d(x_{i+1}, x_i) \leq d(x_1, x_0) \sum_{i=m}^{n-1} (\frac{K}{2})^i$

The geometric series converges because $0 < k < 1$. Therefore, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, sequence $\{x_n\}$ converges to some $x^* \in X$.

Now we show that x^* is a fixed point:

Since, F is continuous (inherited from the contraction property),

$x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} F(x_{n-1}) = F(\lim_{n\to\infty} x_{n-1}) = F(x^*)x$

$x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} F(x_{n-1}) = F(\lim_{n\to\infty} x_{n-1}) = F(x^*)$.

Now Uniqueness:

Suppose x^* and y^* are both fixed points. Then,

$d(x^*, y^*) = d(Fx^*, Fy^*) \le \frac{K}{2} d(x^*, y^*)$.

Since $0 < \frac{K}{2} < 1$, subtracting gives:

$(1 - \frac{K}{2}) d(x^*, y^*) \le 0 \Rightarrow d(x^*, y^*) = 0 \Rightarrow x^* = y^*$. Hence, F has a unique fixed point.

Theorem: (Coprime iterative sequence fixed point theorem)

Statement: Let (X, d) be a metric space and $F: X\to X$ such that the iteration $x_{n+1} = F(x_n)$ follows an index sequence $\{n_k\}$ where $GCD(n_k, n_{k+1}) = 1$. If sequence $\{x_n\}$ converges, then its limit is a fixed point of F.

Proof: Let $x_{n+1}= F(x_n)$, and assume $\{x_n\}\to x^*$ to $x^*\{x_n\}\to x^*$ as $n\to\infty$.

We want to prove that $F(x^*) = x^*$, That is x^* is a fixed point.

We use the continuity assumption (inferred from convergence under contraction): $F(x^*) = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} x_{n+1} = x^*$.

Role of coprimality:

The coprime condition $GCD(n_k, n_{k+1}) = 1$ ensures that the iteration does not cycle or collapse into a repetitive non-converging loop. It provides diversity in the index evolution, which supports convergence.

Hence, the limit of the coprime-index iteration is a fixed point on f.

Theorem: (Coprime pairwise contraction theorem)

Statement:

Let (X, d) be a complete metric space, and for all distinct $x, y \in X$,

$d(Fx, Fy) \le kd(x, y)$, if $GCD(\phi(x), \phi(y)) = 1$, with $0 < k < 1$.

Thus F has at most one fixed point.

Proof: Assume $x, y \in X$ are two fixed points of F.

Therefore, $F(x) = x$ and $F(y) = y$.

Apply the contraction condition: $d(x, y) = d(Fx, Fy) \le k\, d(x, y)$.

Subtracting: $(1-k)\, d(x, y) \le 0 \Rightarrow d(x, y) = 0 \Rightarrow x = y$.

Therefore, any two fixed points must be equal. Thus the fixed point is unique.

GENERALIZED FIXED POINT RESULTS.

The distance between points and images is compared.

- **Weak contractions:**

Definition: A mapping F is a weak contraction if $d(fx, fy) \leq d(x, y) - \phi(d(x, y))$

Weak contraction in the coprime metric:

Theorem: Let F: N→N be a weak contraction in the coprime metric space. Then F has a unique fixed point.

Proof: construct a sequence $x_{n+1} = Fx_n$

If $d(x_n, x_{n+1}) = 2$. Then $d(x_{n+1}, x_{n+2}) - \phi(2) < 2 \Rightarrow d(x_{n+1}, x_{n+2} \in \{0, 1\}$.

Hence, the sequence stabilizes, yielding a fixed point.

- **Kannan-type mappings**:

Definition: A mapping F:X⟶X satisfies the kannan condition if

$d(Fx, fy) \leq \alpha\ [d(x, Fx) + d(y, yx)$ for all $x, y \in X$ and $0 \leq \alpha < \frac{1}{2}$

Kannan in the coprime metric:

Theorem: Let F: N⟶N satisfy the kannan condition. Then F has a unique fixed point in (N, d).

Proof: construct a sequence $x_{n+1} = Fx_n$

Use the kannan inequality is used to obtain, $d(x_{n+1}, x_{n+2}) \leq \alpha\ [d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})$

Eventually, $x_n = x_{n+1} \Rightarrow Fx_n = x_n$

- **ĆiriC-type mappings:**

Definition: A mapping F: X⟶X satisfies the **ĆiriC-type** condition if

$d(Fx, Fy) \leq \alpha \max\{d(x, y) + d(x,Fx),$

$d(y, Fy)\ d(++x, Fy), d(y, Fx)\}$ for all $0 \leq \alpha < 1$

Ciric in Coprime metric:

Theorem: If F satisfies the Ciric condition in the coprime metric space (N, d), F exhibits a unique fixed point .

Proof: Let $x_0 \in N$, define $x_{n+1} = Fx_n$

The recurrence under Ciric condition implies:

$d(x_{n+1}, x_{n+2}) \leq \alpha M_n$ with $M_n \in \{0,1,2\}$, $\exists\ n_0$ such that,

$d(x_{n0}, x_{n0+1}) = 0 \Rightarrow x_{n0} = Fx_{n0}$

Examples

We define F: N→N by: F(x)= x , and GCD (x, 2), if x is prime otherwise

In this case:

- For primes x, F(x) = x (fixed points).
- For composites x, F(x) =1 or 2.

Thus, primes are fixed points.

Hypothesis

- **H1:** The introduction of coprime conditions introduces sufficient contraction to guarantee the existence and uniqueness of the fixed points.
- **H2:** Using number-theoretic functions such as Euler's totient function or prime labeling on metric points enhances the granularity of contraction conditions.
- **H3:** Coprime-controlled iterations may improve convergence in dynamical systems.

Research Gap

No classical studies directly link co-prime structures with metric spaces for fixed point theorems.

- **Multi-valued mappings** (fixed set points)
- **Random coprime metrics**
- **Applications in cryptography**
- **Lack of Integration with Number Theory:** While fixed point theory has been expanded into probabilistic, fuzzy, and ordered spaces, its fusion with number theory, particularly coprime structures, is limited.
- **No Prior Work on Coprime-controlled Contractions:** There is minimal literature on using GCD-based functions to influence contraction mappings in metric spaces.
- **Insufficient Use of Arithmetic Structures:** Most fixed point theorems do not utilize properties such as prime distributions, Euler functions, or modular congruences
- **Sparse Examples of Coprime Iterations:** Iterative methods rarely include arithmetic constraints, which may offer to improve the convergence guarantees in computational applications.

CONCLUSION

This study introduces a novel class of fixed point theorems in metric spaces using co-primality numbers. By embedding number-theoretic structures within the Banach contraction mapping conditions, we defined coprime controlled mappings and established their convergence and uniqueness properties. These results pave the way for further exploration at the intersection of the fixed-point theory, dynamical systems, and number theory. Future work may involve computational models, fractal geometry, topological interpretations, and applications in encryption and prime-based hashing algorithms.

Acknowledgments: The authors would like to thank the editors and reviewers for their valuable suggestions.

Conflicts of interest: The author declares that he has no conflicts of interest.

REFERENCES

1. Banach, S. (1922). Sur les opérations dans les ensembles abstraits. Fundamenta Mathematicae, V.3, PP.133–181.Kirk, W. A. (1965).
2. A fixed point theorem for mappings which do not increase distances. American Mathematical Monthly,V.72, PP.1004–1006.

3. Jungck, G.(1976). Commuting mappings and fixed points. American Mathematical Monthly,V.83,PP.261–263.
4. Agarwal, R. P. O'Regan, D., and Sahu, D. R. (2009). Fixed point theory for Lipschitzian-type mappings. Springer. pp.1–320.
5. Hardy, G. H. and Wright, E. M. (2008). An introduction to the theory of numbers (6th ed.). Oxford University Press. pp. 19–22, pp.PP. 83–90 (coprime properties and Euclid's algorithm).
6. Khamsi, M. A. and Kirk, W. A. (2001). An Introduction to metric spaces and fixed point theory. John Wiley and Sons. pp. 52–60, pp. PP.87–101.
7. Gopal, D. and Rhoades, B. E. (2011). Fixed point theorems via altering distance functions. Journal of Mathematical Analysis and Applications, V.375(1),PP. 264–272.
8. Smith, M.J.(2014). Coprime structures in metric dynamics. Journal of number theory, V.143, PP 92–106.
9. Altun, I. and Sadarangani, K.(2012). Generalized contractions and fixed point theorems. Nonlinear Analysis, V.75(4), PP.2154–2165.
10. Berinde, V. (2007). Iterative approximation of fixed points. Springer. pp. 101–115, pp. 148–160.
11. Caristi, J. (1976). Fixed point theorems for mappings satisfying inwardness conditions. Transactions of the American Mathematical Society,V.215, PP.241–251.
12. Bruck, R. E. (1973). Nonexpansive projections on subsets of Banach spaces. Pacific Journal of Mathematics,V. 47(2), PP.341–355.
13. Kosmol, P.(1987). Optimization and fixed point theory.Springer.PP.65–80.
14. Dutta, P. N., and Choudhury, B. S. (2008). A generalization of contraction principle in metric spaces. Fixed point theory and applications,2008, Article ID 406368, PP. 1–8.
15. Gregus, M. (1980). A fixed point theorem in complete metric spaces. Mathematica Slovaca,V. 30(2),PP. 107–111.
16. Abkar, A. and Eslamian, E. (2010). Common fixed points for weakly compatible maps in fuzzy metric spaces. Fuzzy Sets and Systems, V.161(4), pp.412–420.
17. .Kirk, W. A. (2003). Fixed point theory in metric spaces. Lecture Notes, University of Iowa. pp. 10–29.
18. Singh, S. P. and Mishra, M. K. (1990). Fixed point theory and applications. Macmillan India. pp. 98–115, pp. 135–142.