INTERNATIONAL JOURNAL OF RESEARCH AND INNOVATION IN APPLIED SCIENCE (IJRIAS)
ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS |Volume XI Issue VI June 2026

Fixed Point Theorems for Reich-Type Cyclic Contraction in Partial Symmetric Spaces

P. K. Eswari

Professor, Department of Mathematics, Mookambigai College of Engineering, Pudukkottai, Tamil Nadu, India

DOI: https://doi.org/10.51584/IJRIAS.2026.11060227

Received: 25 June 2026; Accepted: 30 June 2026; Published: 13 July 2026

ABSTRACT

In this paper, we establish fixed point theorems for Reich-type cyclic contraction mappings in partial symmetric spaces. Under suitable completeness assumptions, we prove the existence and uniqueness of fixed points for such mappings. The obtained results extend and generalize several existing fixed point theorems for cyclic contractions in metric-type spaces. These findings contribute to the theory of partial symmetric spaces and provide a framework for further applications in nonlinear analysis.

Keywords: Fixed point, Reich-type cyclic contraction, partial symmetric space, complete space, Banach contraction principle.

INTRODUCTION

Fixed point theory plays a fundamental role in nonlinear analysis and has extensive applications in optimization, differential equations, engineering, computer science, and economics. The celebrated Banach contraction principle provides a powerful tool for proving the existence and uniqueness of solutions to many mathematical problems.

Several generalizations of metric spaces have been introduced to broaden the scope of fixed point theory. Among these, partial metric spaces introduced by Matthews[9] have attracted considerable attention due to their applications in computer science and domain theory. Later, partial symmetric spaces emerged as a natural extension where self-distances need not vanish while the symmetry property is preserved.

Mathews [9] introduced the concept of partial metric spaces. Combining the ideas involved in the concept of partial metric spaces and symmetric spaces, Mohamed Asim[10] introduced the class of partial symmetric spaces, where in they proved existence and uniqueness fixed point results for certain types of contractions in partial symmetric spaces. There are so many fixed point theorems in metric spaces, symmetric spaces and partial symmetric spaces for contractive mappings[1,2,3,4,6,7]. And the same can be extended in partial symmetric spaces.

The main purpose of this paper is to give fixed point theorems for Reich-type contraction mappings in a partial symmetric space and prove the existence and uniqueness of a fixed point in partial symmetric spaces.

PRELIMINARIES

Definition 1[10]: Let X be a nonempty set. A function $p: X\times X \to [0,\infty)$ is called a partial symmetric if for all $x, y, z \in X$,

1. $x=y$ if and only if $p(x,x) = p(x,y) = p(y,y)$,
2. $p(x, x) \leq p(x, y)$,
3. $p(x, y)=p(y, x)$,

4. $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Then (X,p) is called a partial symmetric space.

Definition 2[10]:

A sequence $\{x_n\}$ in (X, p) converges to $x \in X$ if $\lim_{n\to\infty} p(x_n, x) = p(x, x)$.

Definition 3[10]:

A sequence $\{x_n\}$ in (X, p) is called Cauchy sequence if $\lim_{m,n\to\infty} p(x_m, x_n)$ exists and is finite.

Definition 4[10]:

A partial symmetric space (X, p) is said to be complete if every Cauchy sequence converges to a point in the space.

Definition 5[8]:

Let A and B be nonempty subsets of X. A mapping T: AUB→AUB is called cyclic if $T(A) \subseteq B$, $T(B) \subseteq A$.

Definition 6[5]: Reich-Type Cyclic Contraction

Let (X, p) be a partial symmetric space and let A and B be nonempty subsets of X. A cyclic mapping T: AUB→AUB satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ is called a Reich-Type Cyclic Contraction if there exists constants $a,b,c \geq 0$ such that $a+b+c < 1$ and

$p(Tx,Ty) \leq a\, p(x,y) + b\, p(x,Tx) + c\, p(y,Ty)$ for all $x \in A$ and $y \in B$.

MAIN RESULT

Theorem 1

Let (X,p) be a complete partial symmetric space and let A and B be nonempty closed subsets of X such that X=AUB. Suppose that T:X→X is a cyclic mapping satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ and $p(Tx,Ty) \leq k\, p(x,y)$ for all $x \in A$ and $y \in B$, where $0 < k < 1$.Then T has a unique fixed point $u \in A \cap B$.

Proof

Choose an arbitrary point $x_0 \in A$.

Define $x_{n+1} = Tx_n$

Since T is cyclic, $x_{2n} \in A$ and $x_{2n+1} \in B$.

Applying the contraction condition to x_n and x_{n-1},

$P(x_{n+1}, x_n) = P(Tx_n, Tx_{n-1}) \leq k\, p(x_n, x_{n-1})$

By induction, $P(x_{n+1}, x_n) \leq k^n\, p(x_1, x_0)$.

Using Partial Symmetric Triangle inequality,

For $m > n$, $P(x_m, x_n) \leq \sum_{i=n}^{m-1} k^i p(x_1, x_0)$

Thus,

$$\lim_{n\to\infty} p(x_m, x_n) = 0$$

Hence, $\{x_n\}$ is a Cauchy sequence.

Completeness of X implies the existence of $u \in X$ such that $\lim_{n\to\infty} p(x_n, u) = p(u,u)$. Since even terms belong to A and odd terms belong to B, and both subsets are closed, we obtain $u \in A \cap B$.

Using continuity induced by the contraction condition, Tu=u.

Hence u is a fixed point.

To prove uniqueness, suppose that u and v are fixed points.

Then $p(u,v) = p(Tu,Tv) \leq k\, p(u,v)$.

Thus, $(1-k)\, p(u,v) \leq 0$.

Since $k<1$, $p(u,v)=0$.

By the axioms of partial symmetric spaces, $u = v$.

Therefore, the fixed point is unique.

Theorem 2

Let (X,p) be a complete partial symmetric space and let A and B be nonempty closed subsets of X. Suppose that $T: A\cup B \to A\cup B$ is a Reich -Type Cyclic Contraction mapping satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ and $p(Tx,Ty) \leq a\, p(x,y) + b\, p(x,Tx) + c\, p(y,Ty)$ for every $x \in A$ and $y \in B$, where $a, b, c \geq 0$ such that $a + b + c < 1$.Then T has a unique fixed point $u \in A \cap B$.

Proof

Choose an arbitrary point $x_0 \in A$.

Define $x_{n+1} = Tx_n$

Since T is cyclic, $x_{2n} \in A$ and $x_{2n+1} \in B$.

Applying the contraction condition to x_n and x_{n-1},

$P(x_{n+1},x_n) = P(Tx_n,Tx_{n-1}) \leq a\, p(x_n,x_{n-1}) + b\, p(x_n, x_{n+1}) + c\, p(x_{n-1}, x_n)$

Hence $(1-b)\, p(x_n, x_{n+1}) \leq (a + c)\, p(x_n,x_{n-1})$

Therefore, $p(x_n, x_{n+1}) \leq \lambda\, p(x_n,x_{n-1})$ where $\lambda = (a + c)/ (1-b)$

Since $a + b + c < 1$, $\lambda< 1$

By induction, $P(x_{n+1},x_n) \leq \lambda^n p(x_1,x_0)$.

Using Partial Symmetric Triangle inequality,

For $m > n$, $P(x_m, x_n) \leq \sum_{i=n}^{m-1} \lambda^i p(x_1, x_0)$

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS |Volume XI Issue VI June 2026

Thus,

$$\lim_{n\to\infty} p(x_m, x_n) = 0 .$$

Hence, $\{x_n\}$ is a Cauchy sequence.

Completeness of X implies the existence of $u \in X$ such that $\lim_{n\to\infty} p(x_n, u) = p(u,u)$. Since even terms belong to A and odd terms belong to B, and both subsets are closed, we obtain $u \in A \cap B$.

Using continuity induced by the contraction condition, Tu=u.

Hence u is a fixed point.

To prove uniqueness, suppose that u and v are fixed points.

Then p(u,v) = p(Tu,Tv) ≤ a p(u,v) + b p(u,u) + c p(v,v).

Using p(u,u) ≤ p(u,v) and p(v,v) ≤ p(u,v)

We obtain p(u,v) = p(Tu,Tv) ≤ (a + b + c) p(u,v)

Thus, p(u,v) ≤ p(u,v), since a + b + c < 1.

Therefore, p(u,v) = 0.

By the axioms of partial symmetric spaces, u = v.

Therefore, the fixed point is unique.

EXAMPLES

Example 1:

Let X = [0,1], and define p(x,y) = max{x,y}.

Then (X,p) is a complete partial symmetric space.

Let A = [0,1/2], B = [1/2,1].

Define

$$T(x) = \frac{x}{4} + \frac{1}{2} .$$

Then $T(A) \subseteq B$, $T(B) \subseteq A$.

Moreover, p(Tx,Ty) ≤ 1/4 p(x,y)

Hence T satisfies the contraction condition with k=1/4.

By Theorem 1, T possesses a unique fixed point.

Example 2:

Let X = {0,1}, and define the partial symmetric $p: X\times X \to R_{\geq 0}$ by

p(0,0) = 0, p(1,1)=1, p(0,1) = p(1,0) = 2, This function is:

- Symmetric: $p(x,y)=p(y,x)$.
- Allows nonzero self-distance: $p(x,x)\neq 0$ for $x=1,2$.
- Satisfies the axioms of a partial symmetric space.

Hence (X,p) is a complete partial symmetric space.

Take $A=\{0\}$, $B=\{0,1\}$.Clearly, $X = A\cup B$ and $A\cap B = 0 \neq \phi$

Define the mapping $T: X\rightarrow X$ by $T(0)=0, T(1)=0$.

Then $T(A)\subseteq B$, $T(B)\subseteq A$.

Hence, T is a cyclic mapping.

Suppose there exist constants $a, b, c \geq 0$ with $a+b+c<1$,such that for every $x\in A$, $y\in B$,

$p(Tx,Ty)\leq a\, p(x,y)+b\, p(x,Tx)+c\, p(y,Ty)$.

For example, choose $a=0.25$, $b=0.25$, $c=0.25$, so that $a+b+c = 0.75 <1$.

We verify the only nontrivial case:

For $x = 0$, $y = 1$, $p(T0,T1) = p(0,0) = 0$.

The right-hand side is

$0.25\, p(0,1) + 0.25\, p(0,0) + 0.25\, p(1,0) = 0.25(2) + 0 + 0.25(2) = 1$.

Hence $0 \leq 1$, which is true.

Similarly, $x =1$, $y =0$ also satisfies the inequality. Thus T satisfies the Reich-type cyclic contraction.

Since $T(0)=0$, 0 is a fixed point. Also, $T(1) =0 \neq 1$, so 1 is not a fixed point.

Hence by Theorem 2, the unique fixed point is 0.

CONCLUSION

We have established fixed point theorems for Reich-Type cyclic contraction mappings in complete partial symmetric spaces. The theorem guarantees the existence and uniqueness of a fixed point under Reich-Type cyclic contraction condition. The obtained result generalizes several known fixed point theorems in metric and partial metric settings. Future work may focus on multi-valued cyclic contractions, weak contractions, and applications to integral and differential equations.

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