The Study of One-Dimensional Relativistic Shannon Entropy with Hulthen–Coulomb Interaction Using Klein-Gordon Equation

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ABSTRACT

The study of quantum information entropies is significant in quantum computation and signal analysis. Much research work has been carried out in quantum information entropies under nonrelativistic wave equation. However, not much research has been reported in evaluating quantum information entropies using relativistic wave equation. In this work, we evaluate Shannon information entropy in one-dimensional space with generalised Hulthen plus Coulomb potential (GHPCP) in Klein-Gordon equation using parametric Nikiforov-Uvarov method. The numerical computations obtained both for position and momentum spaces as well as the Shannon sum obeys Bialynicki-Birula and Mycieslski (BBM) uncertainty inequality in order to ascertain the degree of accuracy of the analytical and numerical calculation. This research has relevance in atomic and molecular physics especially in localisation and delocalisation of electrons or particle in atomic orbitals.

Keywords: Klein-Gordon Equation, Nikiforov-Uvarov Method Generalised Hulthen Plus Coulomb potential, Shannon information entropy.

INTRODUCTION

Quantum information entropies in particular the Fisher and Shannon entropies are used as global identifier of uncertainty as well as electron's localisation and delocalisation. Information entropy especially, the Shannon entropy has significant applications to nanosystems which are building block to information communication and technology [1-11] . The significance of Shannon entropy lies in its role as a generalized version of Heisenberg's uncertainty principle, offering a quantification of uncertainty in quantum systems and characterizing other physical properties [2, 12, 13, 14, and 15]. Meanwhile, a lot of research work has been carried out for various quantum information entropies using Schrodinger wave equation. Okon et.al [1] carried out both analytical and computational study of Fisher and Shannon information entropic energy interaction in one and three dimensional spaces for homonuclear and heteronuclear diatomic molecules with exponential-type potential using supersymmetric quantum mechanics approach. Their results of Fisher and Shannon spectral plots showed a strong interaction and localization of homonuclear hydrogen and heteronuclear lithium hydride molecules respectively. Sun et al. [16] evaluated Shannon entropy for a lower position and momentum eigenstates of Poschl–Teller-like potential. The entropic densities were presented graphically while the numerical computations for the Shannon entropy obeyed BBM inequality. Bakke and Furtado [17] investigated the interaction between the Klein-Gordon oscillator and a modified mass term. Here, the authors obtained relativistic bound states for both attractive and repulsive Coulomb-type potential. Meanwhile, a lot of research work has been carried out with generalized Hulthen and Coulomb potential within relativistic and nonrelativistic quantum mechanics using Klein-Gordon and Schrodinger wave equations respectively.

The generalized Hulthen potential is one of the most important short-range exponential-type potentials used both for relativistic and nonrelativistic quantum mechanics. It has attracted attention due to its enormous and significance applications in atomic, nuclear, molecular and plasma physics as well as quantum field theory. This potential behaves similarly to the Coulomb potential near the origin and decreases exponentially at large distances. This feature makes it suitable for describing screened interactions in atomic systems, nuclear systems, and plasma environments [18]. The Schrödinger equation with the generalized Hulthén potential has been extensively investigated since exact and approximate analytical solutions can be obtained for several quantum systems focusing on deriving bound-state energy eigenvalues and wave functions for arbitrary angular momentum states.

Ikhdair and Sever [19] studied approximate eigenvalue and eigen function solutions of the Schrödinger equation with the generalized Hulthén potential using the Nikiforov–Uvarov method. They derived analytical expressions for energy spectra and corresponding eigen functions in terms of Jacobi polynomials. Their study demonstrated that the generalized Hulthén potential effectively describes short-range interactions and provides useful approximations for arbitrary l-states. Qiang and Dong [18] successfully applied supersymmetric quantum mechanics approach (SUSYQM) to derive exact relativistic bound-state solutions for generalized Hulthén potentials. Their work demonstrated that generalized Hulthén systems possess supersymmetric structures that simplify relativistic calculations. The development of PT-symmetric and non-Hermitian quantum mechanics led to renewed interest in generalized Hulthén potentials. Simsek and Egrifes [19] investigated the Klein–Gordon equation with complex generalized Hulthén potentials in PT-symmetric quantum mechanics. Their study demonstrated that non-Hermitian generalized Hulthén systems may still possess real energy spectra under PT-symmetry conditions. Egrifes and Sever [20] further studied PT-symmetric generalized Hulthén potentials using the Nikiforov–Uvarov method. They obtained exact relativistic bound-state solutions and established important relationships between PT symmetry, pseudo-Hermiticity, and exponential-type potentials. Antia and Ikot [21] studied relativistic spinless particles interacting through combined Hulthén and Yukawa potentials using the generalized parametric Nikiforov–Uvarov method. Their study demonstrated that generalized Hulthén-type interactions contain several important special cases including Coulomb and Yukawa potentials. Karayer et .al [22] investigated the local fractional Klein–Gordon equation with generalized Hulthén potentials using conformable fractional calculus. Their study showed that fractional derivatives significantly influence relativistic energy spectra and wave functions. The Coulomb potential is one of the most fundamental interaction potentials in physics and quantum mechanics. It describes the electrostatic interaction between charged particles and plays a central role in atomic physics, molecular physics, nuclear physics, plasma physics, and quantum electrodynamics. A lot of work has been reported in existing literature on the solution of Coulomb potential within the confined of relativistic (Klein-Gordon and Dirac) and nonrelativistic (Schrodinger) wave equations.

One of the most important features of Coulomb potential is its singularity both at the origin and infinite range. The long-range nature of the Coulomb force also modifies scattering states, requiring specialized Coulomb wave functions rather than plane waves. This important property is relevance in atomic collision theory and nuclear reaction modeling [23]. The Coulomb field also serves as a testing ground for relativistic quantum theories and quantum electrodynamics (QED), where electron–photon interactions are treated perturbatively around Coulombic bound states [24]. Modern approaches extend the Coulomb problem beyond wave mechanics. It is worth mentioning that phase-space formulations using Wigner functions provide an alternative representation of Coulomb systems, allowing simultaneous analysis in position and momentum space .These methods are useful in semi-classical analysis and quantum statistical mechanics. Example, Vianna et al. [25] investigates the Coulomb potential using the framework of symplectic quantum mechanics (SQM)**,** also known as quantum mechanics in phase space. Instead of describing quantum systems only in coordinate or momentum space, the authors formulate the Schrödinger equation directly in phase space using the Weyl–Wigner formalism. Also, the authors focus specifically on the two-dimensional hydrogen atom, whose interaction is governed by the Coulomb potential. Meanwhile, the combination of generalized Hulthen and Coulomb potential to investigate one-dimensional Shannon entropy using one-dimensional Klein-Gordon equation is yet to be reported in any of the existing literature which gives the motivation of this work. This study is relevance in information communication and propagation, signal analysis as well as electron's localization and delocalization in atomic orbitals. The generalized Hulthen plus Coulomb potential (GHPCP) in one-dimension is given as

$$S(x) = -\frac{be^{-ax}}{\left(1-qe^{-ax}\right)} - \frac{D}{a} \ , \quad (1)$$

Where b and D represents the coupling constant which is synonymous to the potential depths, q is the deformed parameter while "a" represents the range of the potential. Meanwhile, the position and momentum space-Shannon entropy is given equations (1) and (2) respectively.

$$S_r = -\int_0^{\infty} |\psi(r)|^2 \log|\psi(r)|^2 \, dr, \quad (2)$$

$$S_p = -\int_{-\infty}^{\infty} |\psi(p)|^2 \log|\psi(p)|^2 \, dp, \quad (3)$$

Where $\psi(r)$ and $\psi(p)$ are position and momentum-space wave function respectively. Meanwhile, the momentum space wave function is obtained by taking the Fourier transform of position space wave function.

Review of the Nikiforov-Uvarov (NU) Method.

The NU method is based on reducing the second order linear differential equation to a generalized equation of hyper-geometric-type [26-33]. This method provides exact solutions in terms of special orthogonal functions as well as corresponding energy eigenvalues. The NU method is applicable to both relativistic and non-relativistic wave equations. Using appropriate coordinate transformation $S = S(x)$ the NU equation can be written as

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0 \quad (4)$$

where $\tilde{\tau}(s)$ is a polynomial of degree one while $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials of at most degree two.

The parametric NU differential equation according to Tezcan and Sever [34] is given as

$$\Psi''(s) + \frac{c_1 - c_2 s}{s(1-c_3 s)}\Psi'(s) + \frac{1}{s^2(1-c_3 s)^2}\left[-A_1 s^2 + A_2 s - A_3\right]\Psi(s) = 0 \quad (5)$$

The parametric constants $c_i (i = 4-13)$ are obtained as follows

$$\begin{aligned}
c_4 &= \frac{1}{2}(1-c_1), & c_9 &= c_3 c_7 + c_3^2 c_8 + c_6 \\
c_5 &= \frac{1}{2}(c_2 - 2c_3), & c_{10} &= c_1 + 2c_4 + 2\sqrt{c_8} \\
c_6 &= c_5^2 + A_1, & c_{11} &= c_2 - 2c_5 + 2(\sqrt{c_9} + c_3\sqrt{c_8}) \\
c_7 &= 2c_4 c_5 - A_2, & c_{12} &= c_4 + \sqrt{c_8} \\
c_8 &= c_4^2 + A_3 & c_{13} &= c_5 - \left(\sqrt{c_9} + c_3\sqrt{c_8}\right)
\end{aligned} \quad (6)$$

The parametric energy-eigen equation is given as

$$c_2 n - (2n+1)c_5 + (2n+1)\left(\sqrt{c_9} + c_3\sqrt{c_8}\right) + n(n-1)c_3 + c_7 + 2c_3 c_8 + 2\sqrt{c_8 c_9} = 0. \quad (7)$$

The total wave function is given as

$$\psi_{nl}(s) = N_{nl} s^{c_{12}} (1-c_3 s)^{-c_{12} - \frac{c_{13}}{c_3}} P_n^{\left(c_{10}-1, \ \frac{c_{11}}{c_3} - c_{10} - 1\right)}(1-2c_3 s), \quad (8)$$

Solution of Klein-Gordon Wave Equation with Generalised Hulthen plus Coulomb potential Using NU.

The one-dimensional Klein-Gordon equation for equal scalar and vector potential is given as

$$\psi''(x)+\frac{1}{\hbar^2c^2}\left[\left(E-V(x)\right)^2-\left(mc^2+S(x)\right)^2\right]\psi(x)=0, \qquad (9)$$

Where V(x) and S(x) are vector and scalar potential respectively. It is required that S(x) > V(x) for the existence of bound state. Applying the following unit conditions: $\hbar=c=1,$ equation (9) simplifies to equation (10).

$$\psi''(x)+\left[-S^2(x)-2mS(x)-\left(m^2-E^2\right)\right]\psi(x)=0, \qquad (10)$$

Substituting equation (3) into equation (10) gives

$$\psi''(x)+\left[-\left(\frac{-be^{-ax}}{1-qe^{-ax}}-\frac{D}{a}\right)-2m\left(\frac{-be^{-ax}}{1-qe^{-ax}}-\frac{D}{a}\right)-\left(m^2-E^2\right)\right]\psi(x)=0, \qquad (11)$$

The coordinate transformation $s=e^{-ax}$ gives

$$\frac{d^2\psi(x)}{dx^2}=a^2e^{-ax}\frac{d\psi(z)}{dz}+a^2e^{-2ax}\frac{d^2\psi(z)}{dz^2}, \qquad (12)$$

Substituting equation (12) into equation (11) yield

$$\frac{d^2\psi(z)}{dz^2}+\frac{(1-z)}{z(1-z)}\frac{d\psi(z)}{dz}+\frac{1}{z^2(1-z)^2}\begin{Bmatrix}-\left(\chi_1+\chi_2+\chi_3+\chi_4-\chi_5+\xi^2\right)z^2\\+\left(\chi_2+2\chi_3+\chi_4-2\chi_5+2\xi^2\right)z-\left(\chi_3-\chi_5+\xi^2\right)\end{Bmatrix}\psi(z)=0, \qquad (13)$$

where

$$\xi^2=\frac{1}{a^2}\left(m^2-E^2\right),\quad \chi_1=\frac{b^2}{a^2},\quad \chi_2=\frac{2bD}{a^3},\quad \chi_3=\frac{D^2}{a^4},\quad \chi_4=\frac{2mb}{a^2},\chi_5=\frac{2mD}{a^3}. \qquad (14)$$

Comparing equation (13) with the standard parametric NU differential equation (5), then, the following parameters are obtained as follows:

$$A_1=\left(\chi_1+\chi_2+\chi_3+\chi_4-\chi_5+\xi^2\right),\quad A_2=\left(\chi_2+2\chi_3+\chi_4-2\chi_5+2\xi^2\right),A_3=\left(\chi_3-\chi_5+\xi^2\right)\quad , \qquad (15)$$

$$c_1=c_2=c_3=1,c_4=0,c_5=-\frac{1}{2},\qquad c_6=\frac{1}{4}+\chi_1+\chi_2+\chi_3+\chi_4-\chi_5+\xi^2$$

$$c_7=-\chi_2-2\chi_3-\chi_4+2\chi_5-2q\xi^2,c_8=\chi_3-\chi_5+\xi^2,\qquad c_9=\frac{1}{4}+\chi_1,$$

$$c_{10}=1+2\sqrt{\chi_3-\chi_5+\xi^2},c_{11}=2+\sqrt{1+4\chi_1}+2\sqrt{\chi_3-\chi_5+\xi^2},$$

$$c_{12}=\sqrt{\chi_3-\chi_5+\xi^2},c_{13}=-\frac{1}{2}-\left[\frac{1}{2}\sqrt{1+4\chi_1}+\sqrt{\chi_3-\chi_5+\xi^2}\right]$$

(16)

Substituting equation (16) into equation (7) and simplifying gives the energy eigen equation for one-dimensional Klein-Gordon equation as

$$E^2-m^2=-a^2\left[\frac{\left(n^2+n+\frac{1}{2}\right)+\left(n+\frac{1}{2}\right)\sqrt{1+\frac{4b^2}{a^2}-\frac{2bD}{a^3}-\frac{2mb}{a^2}}}{(2n+1)+\sqrt{1+\frac{4b^2}{a^2}}}\right]^2+\frac{D^2}{a^2}-\frac{2mD}{a} \qquad (17)$$

Using equation (12), the total un-normalised wave function is given as

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$$\psi_{nl}(s) = N_{nl} s^{\beta} (1-s)^{\frac{1}{2}+\eta} P_n^{(2\beta,2\eta)}(1-2s), \tag{18}$$

Where

$$\beta = \sqrt{\frac{D^2}{a^4} - \frac{2mD}{a^3} - \frac{(E^2-m^2)}{a^2}}, \quad \eta = \frac{1}{2}\sqrt{1+\frac{4b^2}{a^2}} \tag{19}$$

By using Mathematica software, the ground state normalized wave function is given as

$$\psi_{00}(a,x) = \sqrt{\frac{a\Gamma(2\beta+2\eta+2)}{\Gamma(2\beta)\Gamma(2+2\eta)}} (e^{-ax})^{\beta} (1-e^{-ax})^{\frac{1}{2}+\eta} P_0^{(2\beta,2\eta)}(1-2e^{-ax}), \tag{20}$$

Since the Jacobi term $P_0^{(2\beta,2\eta)}(1-2e^{-ax})$ is unity, then the normalized ground state wave function in one dimension is given as

$$\psi_{00}(a,x) = \sqrt{\frac{a\Gamma(2\beta+2\eta+2)}{\Gamma(2\beta)\Gamma(2+2\eta)}} (e^{-ax})^{\beta} (1-e^{-ax})^{\frac{1}{2}+\eta}, \tag{21}$$

1. **Shannon Information Entropy in One-Dimensional Space.**

By substituting equation (20) into equation (2) and with the help of Mathematica software, we obtained position Shannon entropy in one-dimensional space as

$$S_r = -\int_0^{\infty} \left[\sqrt{\frac{a\Gamma(2\beta+2\eta+2)}{\Gamma(2\beta)\Gamma(2+2\eta)}} (e^{-ax})^{\beta} (1-e^{-ax})^{\frac{1}{2}+\eta}\right]^2 \log\left[\sqrt{\frac{a\Gamma(2\beta+2\eta+2)}{\Gamma(2\beta)\Gamma(2+2\eta)}} (e^{-ax})^{\beta} (1-e^{-ax})^{\frac{1}{2}+\eta}\right]^2 dx, \tag{22}$$

Equation (22) can further be simplified as

$$S_r = -\left(\frac{a\Gamma(2\beta+2\eta+2)}{\Gamma(2\beta)\Gamma(2+2\eta)}\right)^2 \int_0^{\infty} \left[(e^{-ax})^{2\beta} (1-e^{-ax})^{1+2\eta}\right] \log\left[(e^{-ax})^{2\beta} (1-e^{-ax})^{1+2\eta}\right] dx, \tag{23}$$

Equation (23) is evaluated numerically using Mathematica software to obtain one-dimensional Shannon information entropy in position space.

The momentum space wave function is obtained by taking the Fourier transform of position-space wave function. The Fourier transform equation is given as

$$\psi_{00}(p) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \psi_{00}(x) e^{-ipx} dx, \tag{24}$$

The momentum space wave function obtained in equation (24) is substituted to equation (3) and evaluated numerically using Mathematica software to obtain moment-space Shannon entropy in one-dimensional space.

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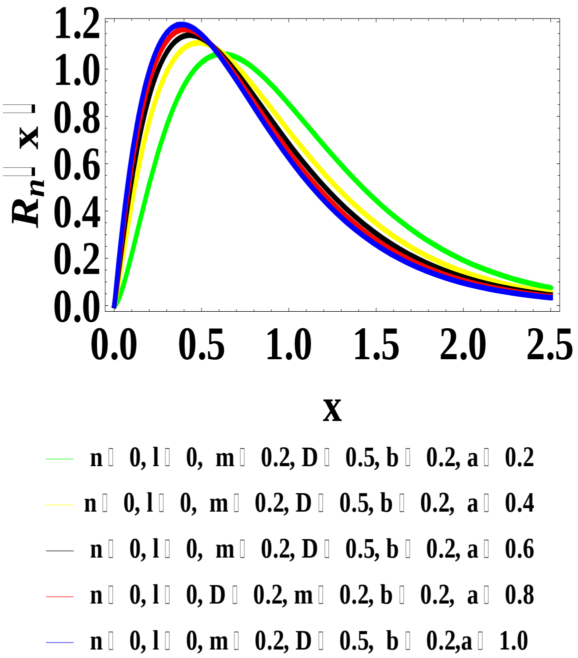

Figure 1(a): Ground state radial wave function plot for varying potential range (a)

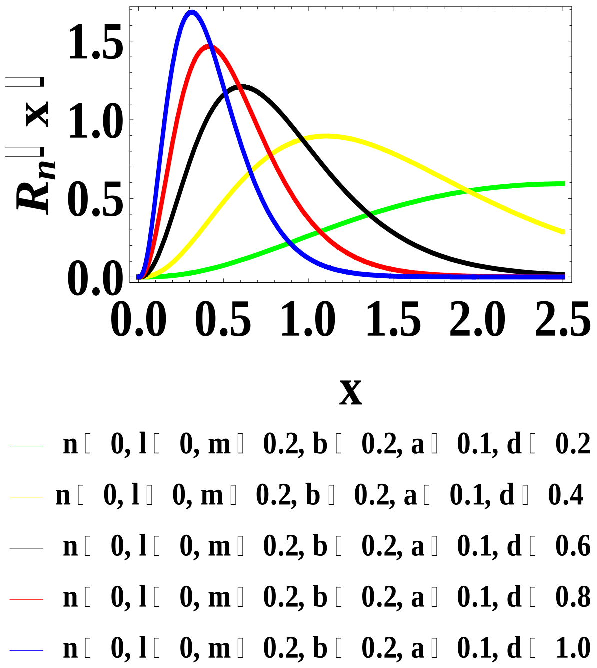

Figure 1(b): Ground state radial wave function plot for varying potential coupling constant (d)

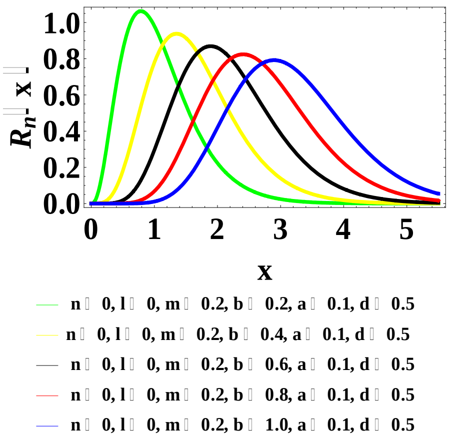

Figure 1(c): Ground state radial wave function plot for varying potential coupling constant (b)

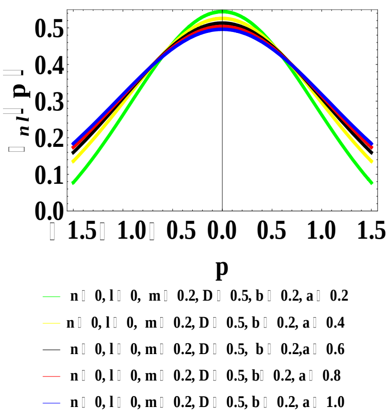

Figure 2(a): Ground state momentum space plot for varying potential range (a)

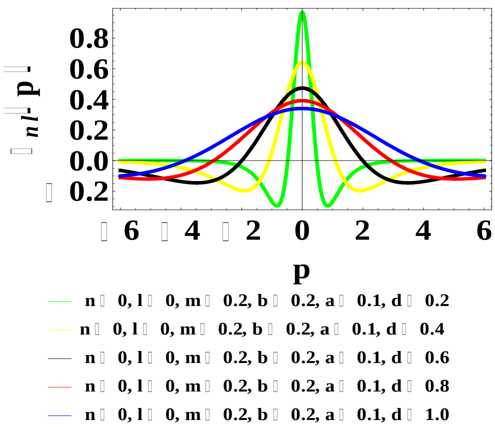

Figure 2(b): Ground state momentum space plot for varying potential coupling constant (d)

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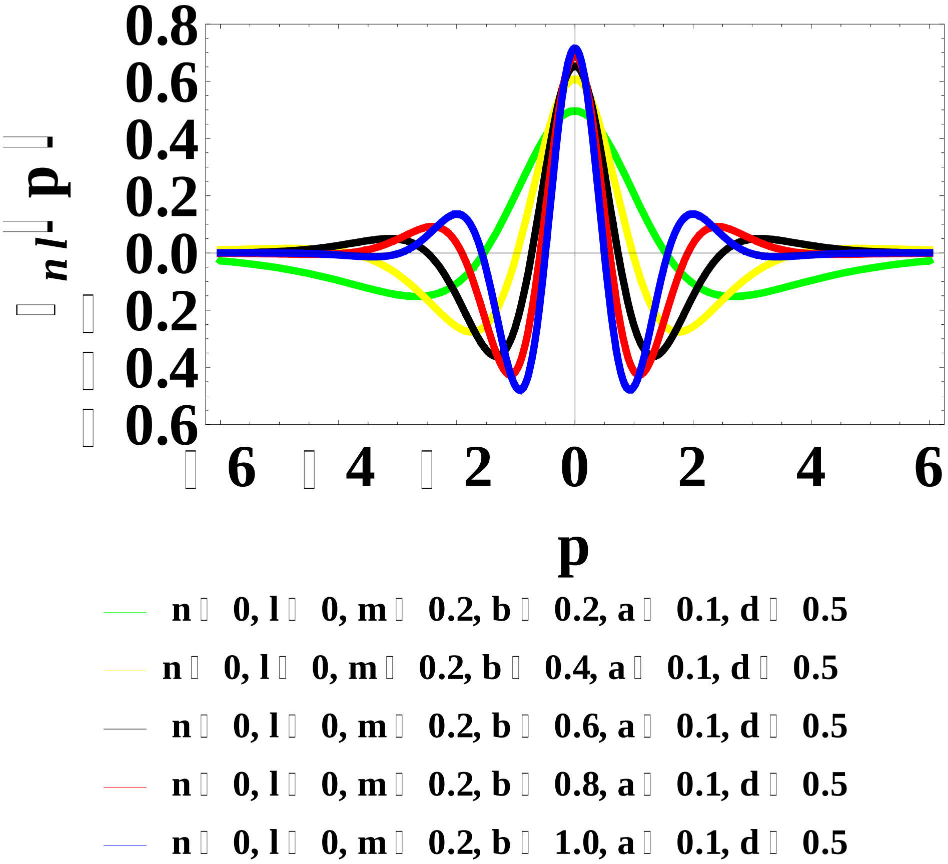


Figure 2(c): Ground state momentum space plot for varying potential coupling constant (a)

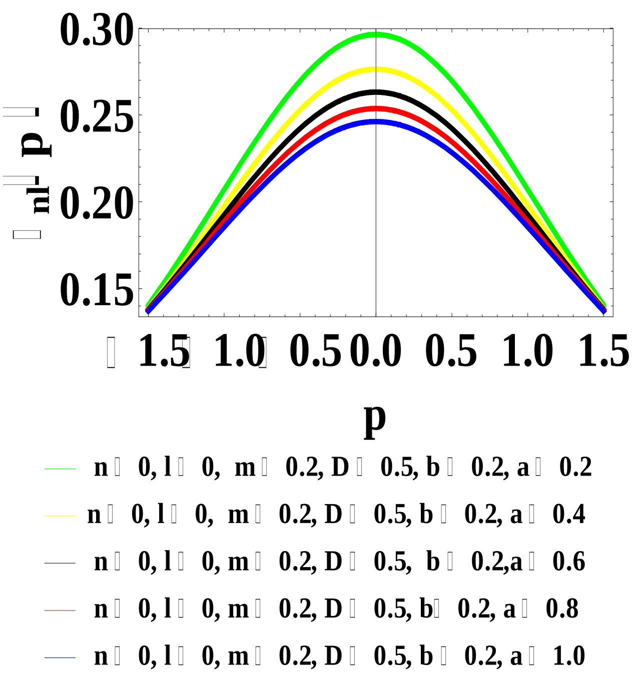


Figure 3(a): Momentum space probability density plot for varying potential range (a)

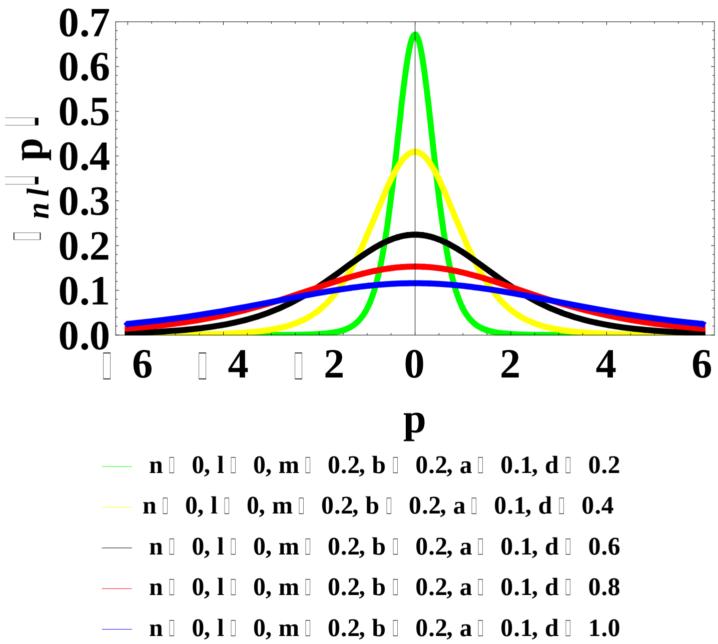


Figure 3(b): Momentum space probability density plot for varying potential coupling constant (d)

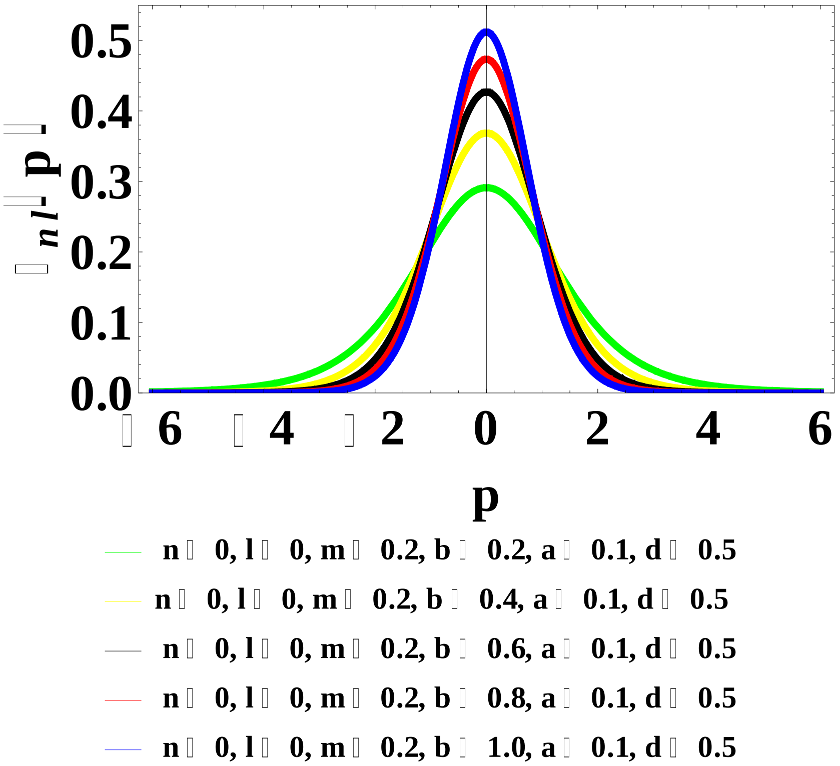
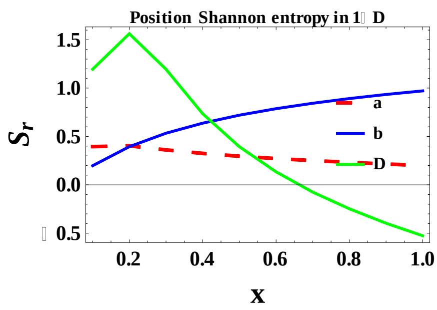


Figure 3(c): Momentum space probability density plot for varying potential coupling constant (a)



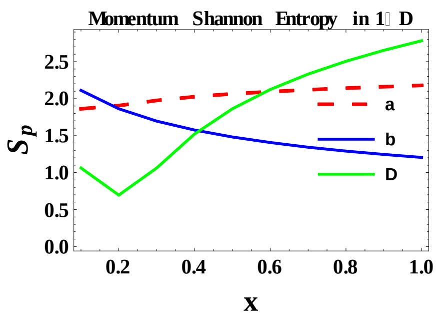


Figure 4(a): The plot of position-space Shannon entropy in one dimensional space

Figure 4(b): The plot of momentum-space Shannon entropy in one dimensional space.

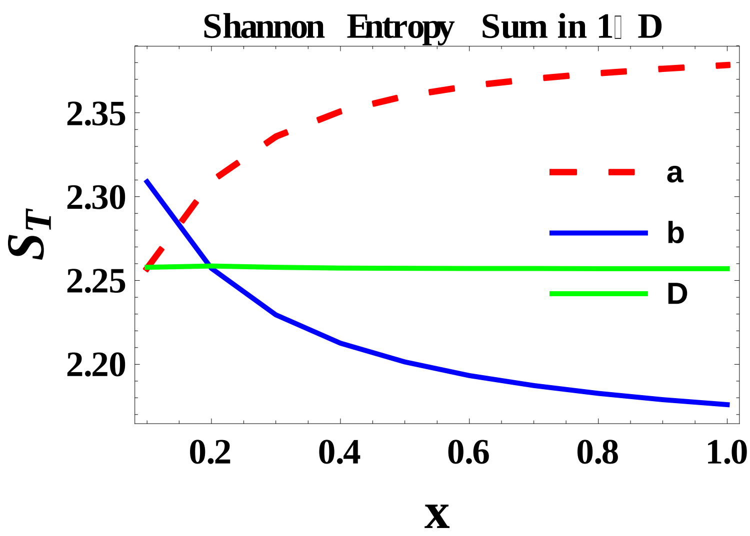


Figure 4(c): The plot of Shannon entropy sum in one dimensional space

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Table 1: Numerical results for relativistic Shannon entropy in one- dimension for variation of potential range (a).

$(S_{T=}S_r + S_p \geq d(1 + \ln\pi) \geq 2.144, b = 0.2, D = 0.5)$

a	S_r	S_p	$S_{T=}S_r + S_p$
0.1	0.39480834865473546	1.8624463818931938	2.2572547305479294
0.2	0.40311664125672420	1.9061437344307397	2.3092603756874640
0.3	0.36095951837705725	1.9749262637208127	2.3358857820978700
0.4	0.32474894796104100	2.0260859900195847	2.3508349379806255
0.5	0.29572443279042180	2.0642525715670783	2.3599770043575000
0.6	0.27162616476366770	2.0944053398018240	2.3660315045654916
0.7	0.25080586445454645	2.1195408897409065	2.3703467541954530
0.8	0.23225662684836285	2.1413764980163830	2.3736331248647460
0.9	0.21536692635806148	2.1609186901797925	2.3762856165378540
1.0	0.19975328239976780	2.1787811502955250	2.3785344326952930

Table 2: Numerical results for relativistic Shannon entropy in one- dimension for variation in Hulthen's coupling parameter (b). $(S_{T=}S_r + S_p \geq d(1 + \ln\pi) \geq 2.144, a = 0.1, D = 0.5)$

b	S_r	S_p	$S_{T=}S_r + S_p$
0.1	0.19790088174152654	2.1109486048446220	2.3088494865861486
0.2	0.39480834865473546	1.8624463818931685	2.2572547305479040
0.3	0.53361768297801040	1.6958955406476584	2.2295132236256690
0.4	0.63778334495562360	1.5748486458139306	2.2126319907695544
0.5	0.72001774817096220	1.4813368173604626	2.2013545655314246
0.6	0.78729918291240110	1.4060125536195665	2.1933117365319674
0.7	0.84379470962512560	1.3435020437673455	2.1872967533924710
0.8	0.89217407604465910	1.2904599748760488	2.1826340509207080
0.9	0.93424435705056230	1.2446726753761033	2.1789170324266656
1.0	0.97128276782322410	1.2046039412192160	2.1758867090424400

Table 3: Numerical results for relativistic Shannon entropy in one- dimension for variation in Coulomb's coupling parameter (D).

$(S_{T=}S_r + S_p \geq d(1 + \ln\pi) \geq 2.144, a = 0.1, b = 0.5)$

D	S_r	S_p	$S_{T=}S_r + S_p$
0.1	1.1948617192701770	1.0630292690191280	2.2578909882893052
0.2	1.5628535375361140	0.6958068157438213	2.2586603532799350
0.3	1.1948617192701745	1.0630292690191272	2.2578909882893017
0.4	0.7359272639855260	1.5215036971501807	2.2574309611357070

0.5	0.3948083486547355	1.8624463818931685	2.2572547305479040
0.6	0.1342430351359809	2.1229230657259097	2.2571661008618906
0.7	−0.0745855539927092	2.3316971913399365	2.2571116373472275
0.8	−0.2482497271468103	2.5053233435951350	2.2570736164483250
0.9	−0.3966708589621479	2.6537156420935270	2.2570447831313790
1.0	−0.5261606871770502	2.7831823501907014	2.2570216630136510

Table 4: Numerical bound state solutions of Klein-Gordon equation in one- dimension for variation in Coulomb's coupling parameter (D). (m = 0.2, n =0, l = 0, a = 0.1)

D	$E_{nl}^{-}(eV)(n=0,l=0)$	$E_{nl}^{+}(eV)(n=0,l=0)$
0.5	-4.799989583	4.799989583
1.0	-9.799994898	9.799994898
1.5	-14.79999662	14.79999662
2.0	-19.79999747	19.79999747
2.5	-24.79999798	24.79999798
3.0	-29.79999832	29.79999832

NUMERICAL RESULTS AND DISCUSSION

Using equation (17), we obtained ground state numerical solutions for one-dimensional Klein-Gordon equation for orbital angular quantum number ($n = l = 0$). This gives quadratic values of 4.799989583 and - 4.799989583 as shown in table 4. In order to compute the relativistic Shannon information entropy, we choose the bound state energy value of - 4.799989583 throughout the numerical computation. Our choice of negative energy is because of the bound state condition which usually occurs at energy less than zero $(E < 0)$. Table 1 is the numerical results for relativistic Shannon entropy in one dimension for variation of potential range (a). Here, the position space entropy decreases with an increase in potential range parameter while the momentum entropy increases with an increase in potential range (a). The Shannon sum obeys BBM inequality which ascertains the degree of accuracy of our analytical and numerical calculations. Table 2 is the numerical results for relativistic Shannon entropy in one dimension for the variation in Hulthen's coupling parameter (b). Here, the position space entropy increases with an increase in the coupling parameter while the momentum space decreases with an increase in the coupling parameter. The Shannon sum also obeys the BBM inequality. Table 3 is the numerical results for relativistic Shannon entropy in one dimension for variation in Coulomb's coupling parameter (D) . Here, the position Shannon entropy decreases with increasing values of coupling parameter (D) while the momentum entropy increases with an increase in the coupling parameter and the Shannon sum obeys the BBM inequality.

Figures 1 (a), (b) and (c) are ground state radial wave function plots for variation in potential range (a) , Hulthen's coupling parameter (b) and Coulomb's coupling parameter (D) respectively. Here the wave functions are all periodic and sinusoidal but increases with the dimension (x). This shows that signal propagation follows periodic and sinusoidal wave form as confirmed experimentally. Figures 2(a), (b) and (c) are the ground state momentum space wave function for varying potential range, Hulthen's coupling constant and Coulomb's coupling constants respectively. In figure 2(a), the momentum space plot are parabolic curves with unique maximum turning point representing the momentarily delocalization of particles away from the turning point in both direction. Figures 2(b) and 2(c) show symmetric property where the wave function is symmetric at both halves from the origin at point (p). Here the symmetric nature indicates equal wave propagation or signals at the same rate in both directions. Figures 3(a), (b) and (c) are momentum space probability density plots for varying potential range, Hulthen's coupling constant and Coulomb's coupling constants respectively. Figure 3(a) is a parabolic

curve with unique maximum turning point. At the maximum point, the particle's localization in momentarily before delocalizing in both direction away from the maximum turning point. Figures 3(b) and 3(c) all follow Gaussian distribution curves with distinct peaks for different values of the variation parameter. Here particle's localization exist at different distinctive peak level. Figures 4(a), 4(b) and 4(c) are plots of position space Shannon entropy, momentum space Shannon entropy and entropy sum in one dimensional space. Figures 4(a) and 4(b) are nonlinear plots that increase with an increase in the dimensional parameter (x) while figure 4(c) is an exponential plot that increases in both directions.

CONCLUSION

In this work we examined and calculate relativistic one dimensional Shannon entropy using one dimensional Klein-Gordon equation. The parametric Nikiforov-Uvarov method was used to obtain the energy eigen equation as well as the total un-normalised wave function. The wave function was then normalized and use to analytically and numerically calculate the relativistic position space, momentum space and Shannon sum information entropies in one dimensional space . The result obtain satisfies the BBM inequality to confirm the suitability of the considerable potential (GHPCP) for the calculation of information theoretic measures using relativistic wave equation. The results obtain is relevance in atomic and molecular physics especially in the description of electrons or particle's localization and delocalization within the atomic orbitals. All plots and numerical calculations were carried out using Mathematica software.

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