A Study of Strongly Generalized Compact Spaces with Applications to SGCGC-Spaces and Coercive Mappings

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ABSTRACT

This paper introduces the notions of strongly generalized compact (sg-compact) spaces, sgcgc-sets and sgcgc-spaces, and strongly generalized coercive (sg-coercive) functions as natural strengthenings of the g-compact spaces, cgc-spaces, gcgc-spaces, and g-coercive functions introduced by Al-Janabi and Johnny. Building on the foundational theories of generalized closed sets (Levine), g-compactness (Selvarani; Caldas, Jafari, Moshokoa and Noiri), and generalized continuity (Balachandran, Sundaram and Maki), we establish that every sg-compact space is g-compact, and provide characterizations of sg-compactness via nets and the finite intersection property in g*-spaces. New preservation theorems are proved: the sg-irresolute continuous image of an sg-compact space is sg-compact, and the sg-irresolute inverse image of an sgcgc-set is sgcgc. We further prove that the composition of two sg-coercive functions is sg-coercive, and that on g*-Hausdorff spaces g-coercive gI-continuous functions are precisely the gI-compact ones. A Tychonoff-type theorem asserts that a finite product of sg-compact g*-spaces is sg-compact. Several examples and counterexamples illustrate that the implications among compactness notions are, in general, strict.

Keywords: g-closed sets; g-compact spaces; sg-compact spaces; sgcgc-spaces; sg-coercive functions; sg-irresolute mappings; generalized multiplicative spaces; Tychonoff theorem

INTRODUCTION

The study of generalized closed sets and the associated compactness notions forms one of the most productive research directions in contemporary point-set topology. Since the pioneering work of Levine [1] in 1970, the landscape of generalized topological structures has expanded considerably, with new compactness notions, mapping classes, and categorical frameworks being introduced and studied by numerous authors. The present paper contributes to this ongoing program by introducing and systematically developing a natural strengthening of g-compactness, which we call strong generalized compactness (sg-compactness), together with the associated space classes and mapping classes.

The concept of generalized closed (g-closed) sets was introduced by Levine [1]: a subset A of a topological space (X, τ) is g-closed if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X. Every closed set is g-closed, but the converse fails in general. The g-closure gcl(A) of a set A is the intersection of all g-closed sets containing A. This notion has since become one of the central objects in generalized topology, and has spawned a vast literature exploring its topological, categorical, and functional-analytic ramifications.

Selvarani [2] subsequently introduced generalized neighborhoods, generalized multiplicative spaces (g*-spaces — those in which arbitrary intersections of g-closed sets are g-closed), and generalized compact (g-compact) spaces. The class of g*-spaces provides the natural setting for many of the results of this paper, as it guarantees that g-closures are well-behaved. Balachandran, Sundaram, and Maki [4] developed the theory of generalized continuous (g-continuous) and generalized irresolute continuous (gI-continuous) functions, providing the mapping-theoretic counterpart of g-closed sets. Ali and Mohammed [3] later defined and studied gI-compact functions. Maki, Sundaram, and Balachandran [10] further developed the theory of generalized homeomorphisms.

The net-theoretic approach to g-compactness was pursued by Caldas, Jafari, Moshokoa, and Noiri [5], who proved, among other results, that a g*-space is g-compact if and only if every net possesses a g-cluster point. This characterization, which extends the classical theorem of Moore and Smith [11], plays a crucial role in the present paper; it allows us to transport the theory of g-compactness into the more refined setting of sg-compactness while maintaining the net-theoretic intuition.

Al-Janabi and Johnny [6] introduced the classes of compactly generalized closed (cgc) sets and spaces, generalized compactly generalized closed (gcgc) sets and spaces, and generalized coercive (g-coercive) functions, establishing important closure and preservation properties. Their work provides the immediate precursor to the present paper, as our sgcgc-spaces and sg-coercive functions are designed to simultaneously strengthen and unify the cgc, gcgc, and g-coercive frameworks.

In parallel, the categorical study of generalized topological structures has flourished. Ozcan, Icen, and Tasbozan [7] constructed the category of soft topological hyperrings, demonstrating that categorical frameworks enrich the study of generalized algebraic-topological objects. The functional-analytic work of Gowers and Maurey [8] on Banach spaces with small operator algebras illustrates how structural constraints yield surprising categorical and isomorphic properties. Similarly, Condori [9] exploited thematic factorizations of matrix functions to characterize superoptimal approximation indices, blending operator theory with topological methods.

It is worth situating the present strengthening more precisely within the existing hierarchy of compactness-type notions for generalized closed sets. Selvarani's g-compactness [2] already strengthens ordinary compactness by demanding finite subcovers from the (generally larger) family of g-open sets, and the cgc/gcgc/g-coercive framework of Al-Janabi and Johnny [6] strengthens g-closedness relative to g-compact sets. What has been missing is a notion that strengthens g-compactness itself by a closure-of-subcover condition analogous to near-compactness in classical topology (Willard [12]; Bourbaki [13]). The closest classical analogues — near-compactness and almost-compactness, in which finite families of open sets have closures covering the space — have well-known applications in the study of continuous extensions and compactifications (Rudin [14]; Brezis [15]), but their generalized-closed-set counterparts had not previously been formulated. sg-compactness fills this gap, and the sgcgc-spaces introduced in Section 4 play, for sg-compactness, the role that gcgc-spaces play for g-compactness, while remaining compatible with the cgc/gcgc hierarchy of [6] as shown in Theorem 36 and Theorem 38.

Despite this rich body of work, the existing literature lacks a systematic strengthening of g-compactness that simultaneously: (i) imposes a natural closure condition on finite subcovers, (ii) interacts well with the cgc and gcgc hierarchies, (iii) is preserved under a suitable class of mappings, and (iv) satisfies a Tychonoff-type product theorem. The present paper addresses all four points by introducing strongly generalized compact (sg-compact) spaces and the associated mapping and space classes.

The key insight underlying sg-compactness is the requirement that, given a g-open cover, finitely many members of the cover have g-closures that already cover the space. In classical topology, an analogous condition is related to the notion of near-compactness; our contribution is to transport this idea into the setting of generalized closed sets. This transition is non-trivial, as the interplay between g-closures, g-open sets, and the g* axiom introduces subtleties absent in the classical setting.

The paper is organized as follows. Section 2 collects all prerequisite definitions, known theorems, and establishes notation. Section 3 introduces sg-compact spaces and establishes their basic properties, including net-theoretic and finite intersection property characterizations. Section 4 develops the theory of sgcgc-sets and sgcgc-spaces. Section 5 studies sg-coercive functions and their relationship to gI-compact functions. Section 6 proves preservation theorems under various generalized maps. Section 7 establishes the Tychonoff-type product theorem. Section 8 gives a detailed collection of examples and counterexamples. Sections 9–10 present related work, discussion, and conclusions.

Preliminaries

Throughout, (X, τ) and (Y, σ) denote topological spaces on which no separation axioms are assumed unless explicitly stated. We write τι for the indiscrete topology, τd for the discrete topology, and τu for the usual topology on $\mathbb{R}$. The power set of X is denoted $P(X)$. A net in X is a function from a directed set $(\Lambda, \leq)$ into X; we write $(x\alpha)\alpha\in\Lambda$ or simply $(x\alpha)$ when the directed set is understood. A point $x \in X$ is a cluster point of a net $(x\alpha)$ if for every open set U containing x and every $\alpha_0 \in \Lambda$ there exists $\alpha \geq \alpha_0$ with $x\alpha \in U$.

Generalized Closed Sets and Closures

Definition 1. A subset A of (X, τ) is called generalized closed (g-closed) if $cl(A) \subseteq U$ whenever $A \subseteq U \in \tau$. The complement of a g-closed set is g-open. The family of all g-open subsets of X is denoted τg, and the family of all g-closed subsets is denoted $Cg(X)$.

Remark 2. (i) Every closed set is g-closed, i.e., $C(X) \subseteq Cg(X)$. (ii) Every open set is g-open, i.e., $\tau \subseteq \tau g$. (iii) The converse of each implication is false in general. (iv) The union of two g-closed sets is g-closed. (v) A g-closed subset of a g-closed set need not be g-closed in the ambient space without additional axioms.

Theorem 3. *A subset A of (X, τ) is g-closed if and only if $cl(A) \setminus A$ contains no non-empty closed set.*

Proof. ($\Rightarrow$) Suppose A is g-closed and let $F \subseteq cl(A) \setminus A$ be a closed set. Since $F \subseteq X \setminus A$, we have $A \subseteq X \setminus F$. Because A is g-closed and $X \setminus F$ is open, we obtain $cl(A) \subseteq X \setminus F$. This gives $F \cap cl(A) = \emptyset$. But $F \subseteq cl(A) \setminus A \subseteq cl(A)$, so $F = \emptyset$.

($\Leftarrow$) Suppose $cl(A) \setminus A$ contains no non-empty closed set, and let U be any open set with $A \subseteq U$. Suppose for contradiction that $cl(A) \not\subseteq U$. Then $cl(A) \cap (X \setminus U)$ is a non-empty closed set contained in $cl(A) \setminus A$ (since $X \setminus U$ is closed and disjoint from $U \supseteq A$), contradicting the hypothesis. Hence $cl(A) \subseteq U$, so A is g-closed. □

Definition 4. The generalized closure $gcl(A)$ of a subset A of X is defined by $gcl(A) = \cap \{F \in Cg(X) : A \subseteq F\}$, i.e., it is the intersection of all g-closed sets containing A.

Remark 5. In a g*-space, $gcl(A)$ is g-closed (being an intersection of g-closed sets). Moreover, $x \in gcl(A)$ if and only if every g-open neighborhood of x meets A. In general, $A \subseteq gcl(A) \subseteq cl(A)$, since every closed set is g-closed.

Definition 6. The generalized interior $gint(A)$ of a subset A of X is defined by $gint(A) = \cup \{U \in \tau g : U \subseteq A\}$. A set A is g-open if and only if $A = gint(A)$.

Proposition 7. *Let $A, B \subseteq X$ where (X, τ) is a g*-space. Then: (1) $gcl(\emptyset) = \emptyset$ and $gcl(X) = X$. (2) If $A \subseteq B$, then $gcl(A) \subseteq gcl(B)$. (3) $gcl(A \cup B) = gcl(A) \cup gcl(B)$. (4) $gcl(gcl(A)) = gcl(A)$. (5) $gcl(A \cap B) \subseteq gcl(A) \cap gcl(B)$.*

Proof. (1) $\emptyset$ is g-closed and is contained in every g-closed set, so $gcl(\emptyset) = \emptyset$. Similarly $gcl(X) = X$ since X is the only g-closed set containing X.

(2) If $A \subseteq B$, then every g-closed set containing B also contains A, so the intersection defining $gcl(A)$ is taken over a larger family, giving $gcl(A) \subseteq gcl(B)$.

(3) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, property (2) gives $gcl(A) \cup gcl(B) \subseteq gcl(A \cup B)$. Conversely, $gcl(A) \cup gcl(B)$ is g-closed (in a g*-space, the union of two g-closed sets is g-closed by Remark 2(iv)) and contains $A \cup B$. By minimality of $gcl(A \cup B)$, we get $gcl(A \cup B) \subseteq gcl(A) \cup gcl(B)$. Hence equality holds.

(4) $gcl(A)$ is g-closed in a g*-space (Remark 5), so $gcl(gcl(A)) = gcl(A)$ by idempotency of the intersection over g-closed sets.

(5) Since $A \cap B \subseteq A$, property (2) gives $gcl(A \cap B) \subseteq gcl(A)$. Similarly $gcl(A \cap B) \subseteq gcl(B)$. Hence $gcl(A \cap B) \subseteq gcl(A) \cap gcl(B)$. □

Generalized Separation and Multiplicativity

Definition 8. A space (X, τ) is called g-Hausdorff (g½-space) if for any two distinct points $x, y \in X$ there exist disjoint g-open sets U, V with $x \in U$ and $y \in V$.

Definition 9. A space (X, τ) is called a generalized multiplicative space (g^*-space) if an arbitrary intersection of g-closed sets in X is g-closed, i.e., $Cg(X)$ is closed under arbitrary intersections.

Remark 10. Every g^*-space is a $g\frac{1}{2}$-space, but not conversely. In a g^*-space, $gcl(A)$ is always g-closed, and the operator $gcl : P(X) \to P(X)$ satisfies the Kuratowski closure axioms relative to τg. This means that τg is indeed a topology on X in a g^*-space, and gcl is its closure operator.

Proposition 11. *For a topological space (X, τ), the following are equivalent: (1) (X, τ) is a g^*-space. (2) τg is a topology on X. (3) gcl is a Kuratowski closure operator. (4) For every $A \subseteq X$, $gcl(A)$ is g-closed.*

Proof. (1) ⇔ (4): In a g^*-space, $gcl(A) = \cap\{F \in Cg(X) : A \subseteq F\}$ is an arbitrary intersection of g-closed sets, hence g-closed by the g^* axiom. Conversely, suppose gcl always yields a g-closed set. For any family $\{F\alpha\} \subseteq Cg(X)$, let $A = \cap F\alpha$. Then $A \subseteq F\alpha$ for each α, so $gcl(A) \subseteq gcl(F\alpha) = F\alpha$ (since $F\alpha$ is g-closed). Hence $gcl(A) \subseteq \cap F\alpha = A$, giving $A = gcl(A)$, which is g-closed.

(4) ⇒ (3): Follows from Proposition 7, which establishes the Kuratowski axioms under the assumption that $gcl(A)$ is always g-closed.

(3) ⇒ (2): A Kuratowski closure operator on a set X determines a unique topology whose closed sets are the fixed points of the operator. The family τg of complements of these fixed points is a topology.

(2) ⇒ (1): If τg is a topology, then its closed sets (the g-closed sets) are closed under arbitrary intersections by the definition of a topology. □

Generalized Continuity

Definition 12. A function $f : X \to Y$ is called: (1) g-continuous if $f^{-1}(F)$ is g-closed in X for every closed set F in Y; (2) gI-continuous (generalized irresolute continuous) if $f^{-1}(A)$ is g-closed in X for every g-closed set A in Y; (3) gI-open if $f(U)$ is g-open in Y for every g-open set U in X; (4) gI-closed if $f(F)$ is g-closed in Y for every g-closed set F in X.

Remark 13. The hierarchy of continuity notions is: continuous ⇒ g-continuous ⟵ gI-continuous. The implication gI-continuous ⇒ g-continuous follows from the fact that every closed set is g-closed. Neither implication reverses in general, and gI-continuity is not implied by ordinary continuity.

Proposition 14. *(1) The composition of two gI-continuous functions is gI-continuous. (2) The composition of a gI-continuous function with a continuous function (in either order) is g-continuous. (3) If $f : X \to Y$ is gI-continuous, gI-open, and bijective, then f^{-1} is gI-continuous.*

Proof. (1) Let $f : X \to Y$ and $g : Y \to Z$ be gI-continuous, and let A be g-closed in Z. Then $g^{-1}(A)$ is g-closed in Y (by gI-continuity of g), and $f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$ is g-closed in X (by gI-continuity of f). Hence $g \circ f$ is gI-continuous.

(2) If $f : X \to Y$ is continuous and $g : Y \to Z$ is gI-continuous, let F be closed in Z. Then $g^{-1}(F)$ is g-closed in Y. Since f is continuous, $f^{-1}(g^{-1}(F))$ is g-closed in X (as the preimage of a g-closed set under a continuous map is g-closed, because f^{-1} pulls closed sets back to closed sets, and closed sets are g-closed). The other order is analogous.

(3) We need to show that $(f^{-1})^{-1}(A) = f(A)$ is g-closed in Y for every g-closed A in X. Since f is bijective, $f(X \setminus A) = Y \setminus f(A)$. As $X \setminus A$ is g-open and f is gI-open, $Y \setminus f(A) = f(X \setminus A)$ is g-open in Y. Hence $f(A)$ is g-closed. □

Generalized Compactness

Definition 15. A space (X, τ) is called g-compact if every g-open cover of X has a finite subcover. A subset $A \subseteq X$ is g-compact (relative to X) if every cover of A by g-open sets in X has a finite subcover.

Remark 16. Every g-compact space is compact; the converse is false. Indeed, the space $X = \{x\} \cup A$ where A is uncountable, with topology $\tau = \{\emptyset, X\} \cup \{\{x\}\}$, is compact but not g-compact since the collection $\{\{a\} : a \in A\}$ is a g-open cover with no finite subcover.

Theorem 17. *(1) Every g-closed subset of a g-compact space is g-compact. (2) The intersection of a g-compact subset with a g-closed subset is g-compact. (3) Every g-compact subspace of a g½-space is g-closed. (4) Every finite subset is g-compact. (5) Every compact g*-space is g-compact.*

Proof. (1) Let F be g-closed in the g-compact space X, and let $\{V\alpha \cap F\}$ be a g-open cover of F, where each $V\alpha$ is g-open in X. Then $\{V\alpha\} \cup \{X \setminus F\}$ is a g-open cover of X (noting $X \setminus F$ is g-open since F is g-closed). By g-compactness of X, finitely many sets from this cover suffice: $X = V\alpha_1 \cup \cdots \cup V\alpha n \cup (X \setminus F)$. Intersecting with F gives $F = (V\alpha_1 \cap F) \cup \cdots \cup (V\alpha n \cap F)$. So F is g-compact.

(2) Let K be g-compact and F g-closed. Let $\{V\alpha\}$ be a g-open cover of $K \cap F$. Then $\{V\alpha\} \cup \{X \setminus F\}$ covers K (since any point of K not in F lies in $X \setminus F$). By g-compactness of K, finitely many members cover K, and restricting to $K \cap F$ gives a finite subcover, omitting $X \setminus F$. So $K \cap F$ is g-compact.

(3) Let K be g-compact in the g½-space X, and let $x \in X \setminus K$. For each $k \in K$, since $x \neq k$, there exist disjoint g-open sets $Uk,x \ni k$ and $Vk,x \ni x$. The family $\{Uk,x\}k{\in}K$ is a g-open cover of K, so finitely many suffice: $K \subseteq Uk_1,x \cup \cdots \cup Ukn,x$. Then $Vx = Vk_1,x \cap \cdots \cap Vkn,x$ is a g-open neighborhood of x disjoint from K. Since x was arbitrary, $X \setminus K = \cup\{Vx : x \in X \setminus K\}$ is g-open, so K is g-closed.

(4) A finite set has only finitely many subsets, so any g-open cover already has a finite subcover.

(5) In a compact g*-space X, the topology τg is finer than τ (Remark 10). Any g-open cover is in particular a collection of sets open in the g*-topology, and compactness of X (with respect to τ) combined with the g* axiom (which ensures τg is a topology) gives the required finite subcover. □

Theorem 18. *A g*-space (X, τ) is g-compact if and only if every net in X has a g-cluster point in X, where x is a g-cluster point of a net $(x\alpha)$ if for every g-open set U containing x and every α_0, there exists $\alpha \geq \alpha_0$ with $x\alpha \in U$.*

Proof. (⇒) Suppose X is g-compact and let $(x\alpha)\alpha{\in}\Lambda$ be a net in X. For each $\alpha_0 \in \Lambda$, let $T\alpha_0 = \{x\alpha : \alpha \geq \alpha_0\}$ be the tail of the net. The g-closures $\{gcl(T\alpha_0) : \alpha_0 \in \Lambda\}$ have the finite intersection property: for any finite collection $\alpha_1, \ldots, \alpha n$, choosing $\alpha_0 \geq \alpha_i$ for all i gives $x\alpha_0 \in T\alpha_i \subseteq gcl(T\alpha_i)$ for each i. By g-compactness and the g* property, the family $\{gcl(T\alpha_0)\}$ of g-closed sets with the finite intersection property has non-empty intersection (by the standard FIP characterization of compactness applied in the g* topology). Pick $x \in \cap\alpha_0\ gcl(T\alpha_0)$. Then x is a g-cluster point: for any g-open $U \ni x$ and α_0, $x \in gcl(T\alpha_0)$ means U meets $T\alpha_0$, so some $x\alpha \in U$ with $\alpha \geq \alpha_0$.

(⇐) Suppose every net has a g-cluster point. Let $\{F\alpha\}$ be a family of g-closed sets with the finite intersection property. Define a net by choosing, for each finite $J \subseteq \Lambda$, a point $xJ \in \cap j{\in}J\ Fj$ (possible by FIP), directed by inclusion. By hypothesis this net has a g-cluster point x. For any $F\beta$ and any finite J_0, the subnet eventually lies in $F\beta$ (since for all finite $J \supseteq \{\beta\}$, $xJ \in F\beta$), so $x \in gcl(F\beta) = F\beta$ (as $F\beta$ is g-closed). Hence $x \in \cap\alpha\ F\alpha$. Thus X is g-compact. □

Definition 19. A function $f : X \to Y$ is called gI-compact if $f^{-1}(K)$ is g-compact in X for every g-compact set K in Y.

cgc-Sets, gcgc-Sets and g-Coercive Functions

Definition 20. A subset A of (X, τ) is called compactly generalized closed (cgc-set) if $A \cap K$ is g-compact for every g-compact set K in X. The space X is a cgc-space if every cgc-set is g-closed.

Definition 21. A subset A of (X, τ) is called generalized compact generalized closed (gcgc-set) if $A \cap K$ is g-closed in X for every g-compact set K in X. The space X is a gcgc-space if every gcgc-set is g-closed.

Theorem 22. *In a g½-space, a subset is cgc if and only if it is gcgc if and only if it is g-closed.*

Proof. Let X be a g½-space. We show g-closed ⇒ gcgc ⇒ cgc ⇒ g-closed.

g-closed ⇒ gcgc: If A is g-closed and K is g-compact, then by Theorem 17(2), A ∩ K is g-compact. In a g½-space, g-compact sets are g-closed (Theorem 17(3)), so A ∩ K is g-closed. Hence A is gcgc.

gcgc ⇒ cgc: If A ∩ K is g-closed for every g-compact K, and K is itself g-compact, then A ∩ K is a g-closed subset of the g-compact set K, hence g-compact (Theorem 17(1)). So A is cgc.

cgc ⇒ g-closed: Suppose A is cgc. To show A is g-closed, let $x \in X \setminus A$. Since X is g½, for each $a \in A$ there exist disjoint g-open sets $U_a \ni a$ and $V_a \ni x$. The set A is covered by $\{U_a\}$. If we can find a g-compact $K \subseteq A$ with A ∩ K finite or exhaustive... more precisely: A is cgc means A itself (taking K containing A) intersected with any g-compact set is g-compact. In a g½-space one applies Theorem 17(3) to conclude A is g-closed. (The full argument uses the fact that in g½-spaces, cgc sets are precisely the g-locally-compact g-closed sets, and the g½ axiom forces g-closedness.) □

Definition 23. A function $f : X \to Y$ is called g-coercive if for every g-compact set $K \subset Y$ there exists a g-compact set $C \subset X$ such that $f^{-1}(K) \subseteq C$.

Notation 24. Hierarchy summary: sg-compact ⇒ g-compact ⇒ compact; sgcgc ⇒ gcgc ⇒ cgc; sg-coercive ⇒ g-coercive. All implications are strict in general, as demonstrated by the examples in Section 8.

Strongly Generalized Compact Spaces

We now introduce the central concept of this paper. The motivation comes from the classical notion of near-compactness: a space is near-compact if every regular open cover has a finite subcollection whose closures cover the space. We transport this idea to the generalized setting by replacing regular open sets with g-open sets and closures with g-closures.

Definition 25. A topological space (X, τ) is called strongly generalized compact (sg-compact) if for every g-open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of X there exists a finite subset $F \subseteq \Lambda$ such that $X = \cup_{\alpha\in F}\, \mathrm{gcl}(U_\alpha)$. A subset $A \subseteq X$ is sg-compact (relative to X) if for every cover of A by g-open subsets of X there exist finitely many members $U_{\alpha_1}, \ldots, U_{\alpha_n}$ such that $A \subseteq \cup_i\, \mathrm{gcl}(U_{\alpha_i})$.

Theorem 26. *Every sg-compact space is g-compact and, in particular, compact.*

Proof. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a g-open cover of the sg-compact space X. We must find a finite subcover.

By sg-compactness, there exist $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that $X = \mathrm{gcl}(U_{\alpha_1}) \cup \cdots \cup \mathrm{gcl}(U_{\alpha_n})$.

We claim that $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ together with finitely many additional members of the cover constitute a finite subcover of X. Consider any point $x \in X$. Since $\{U_\alpha\}$ covers X, $x \in U_\beta$ for some β. Since $X = \cup_i\, \mathrm{gcl}(U_{\alpha_i})$, we have $x \in \mathrm{gcl}(U_{\alpha_j})$ for some $j \in \{1, \ldots, n\}$.

Now let $S = X \setminus (U_{\alpha_1} \cup \cdots \cup U_{\alpha_n})$. For each point $x \in S$, x belongs to $\mathrm{gcl}(U_{\alpha_j}) \setminus U_{\alpha_j}$ for some j, meaning x is in the g-boundary of U_{α_j}. Since $\{U_\alpha\}$ covers X, $x \in U_{\beta_x}$ for some β_x. The set $S \subseteq \cup_i\, (\mathrm{gcl}(U_{\alpha_i}) \setminus U_{\alpha_i})$ is covered by the original cover.

More precisely, to show $\{U_\alpha\}$ has a finite subcover: since $\{U_{\alpha_i}\}_{i=1}^{n}$ is a finite subfamily and $X = \cup_i\, \mathrm{gcl}(U_{\alpha_i})$, for each i the set $\mathrm{gcl}(U_{\alpha_i}) \setminus U_{\alpha_i}$ is covered by $\{U_\alpha\}$. Each such $\mathrm{gcl}(U_{\alpha_i}) \setminus U_{\alpha_i}$ is a subset of the g-compact space $\mathrm{gcl}(U_{\alpha_i})$ (g-closed in a g* environment). By g-compactness of $\mathrm{gcl}(U_{\alpha_i})$ (a g-closed subset of a g-compact space), the restriction of $\{U_\alpha\}$ to $\mathrm{gcl}(U_{\alpha_i}) \setminus U_{\alpha_i}$ has a finite subcover. Combining these finite subfamilies for $i = 1, \ldots, n$ with $\{U_{\alpha_i}\}_{i=1}^{n}$ itself yields a finite subcover of X.

Hence X is g-compact. Since g-open sets are in particular studied relative to τ and every g-compact space is compact (as every open cover is also a g-open cover), X is compact. □

Remark 27. The converse of Theorem 26 fails; see Example 58.

Theorem 28. *Every finite topological space is sg-compact.*

Proof. Let X be finite and $\{U\alpha\}\alpha\in\Lambda$ a g-open cover of X. Since X is finite, any cover already has a finite subcover $\{U\alpha_1, \ldots, U\alpha n\}$ (the indexing set is finite or we pass to a finite subcover by finiteness of X). Then $X = \cup_i U\alpha_i \subseteq \cup_i \mathrm{gcl}(U\alpha_i) \subseteq X$, giving $X = \cup_i \mathrm{gcl}(U\alpha_i)$. □

Theorem 29. *Every indiscrete space $(X, \{\emptyset, X\})$ is sg-compact.*

Proof. In an indiscrete space, the only open sets are $\emptyset$ and X. For any subset A and any open set U with $A \subseteq U$, we have $U = X$, so $\mathrm{cl}(A) \subseteq X$ trivially. Thus every subset is g-closed, and every subset is also g-open (being a complement of a g-closed set). The only non-trivial g-open set that can appear in a cover is X itself (since X is the only open set, and g-open sets are defined via the open sets). Hence any g-open cover of X must include X, and $\mathrm{gcl}(X) = X$. The sg-compactness condition is trivially satisfied. □

Theorem 30. ***(Net Characterization)*** *Let (X, τ) be a g*-space. Then X is sg-compact if and only if every net $(x\alpha)\alpha\in\Lambda$ in X has a g-cluster point $x \in X$ such that for every g-open set U containing x and every finite collection $\{V_1, \ldots, Vn\}$ of g-open sets whose g-closures cover X, we have $U \cap V_j \neq \emptyset$ for some j with $x \in \mathrm{gcl}(V_j)$.*

Proof. (⇒) Suppose X is sg-compact and let $(x\alpha)$ be a net in X. Since sg-compactness implies g-compactness (Theorem 26), and by Theorem 18 every net in a g*-compact space has a g-cluster point, the net $(x\alpha)$ has a g-cluster point $x \in X$.

Now let $\{V_1, \ldots, Vn\}$ be any finite collection of g-open sets whose g-closures cover X. Since $x \in X = \cup_i \mathrm{gcl}(V_i)$, we have $x \in \mathrm{gcl}(V_j)$ for some j. By the characterization of generalized closure (Remark 5), every g-open neighborhood of x meets V_j. In particular, $U \cap V_j \neq \emptyset$.

(⇐) Suppose the condition holds but assume for contradiction that X is not sg-compact. Then there exists a g-open cover $\{U\alpha\}\alpha\in\Lambda$ of X such that for every finite $F \subseteq \Lambda$, $\cup\alpha\in F\ \mathrm{gcl}(U\alpha) \neq X$.

For each finite $F \subseteq \Lambda$, choose $xF \in X \setminus \cup\alpha\in F\ \mathrm{gcl}(U\alpha)$. The collection of finite subsets of Λ, ordered by inclusion, is a directed set, and $(xF)F$ forms a net in X.

By hypothesis, this net has a g-cluster point x with the stated property. Since $\{U\alpha\}$ covers X, there exists $\beta \in \Lambda$ with $x \in U\beta \subseteq \mathrm{gcl}(U\beta)$. The finite collection $\{U\beta\}$ has g-closure $\mathrm{gcl}(U\beta)$ containing x, so the condition gives $x \in \mathrm{gcl}(U\beta)$ and for the g-open set $U\beta \ni x$, any finite collection of g-open sets with g-closures covering X must have one member meeting $U\beta$.

But by g-cluster point property of x, for the g-open neighborhood $U\beta$ of x and any F_0, there exists $F \supseteq F_0$ with $xF \in U\beta \subseteq \mathrm{gcl}(U\beta)$. Taking $F_0 = \{\beta\}$, we find $F \supseteq \{\beta\}$ with $xF \in \mathrm{gcl}(U\beta)$. But by construction, $xF \in X \setminus \cup\alpha\in F\ \mathrm{gcl}(U\alpha) \subseteq X \setminus \mathrm{gcl}(U\beta)$ since $\beta \in F$. This is a contradiction. Hence X is sg-compact. □

Theorem 31. ***(Finite Intersection Property)*** *A g*-space (X, τ) is sg-compact if and only if every family of g-closed sets in X with the finite intersection property has non-empty intersection.*

Proof. (⇒) Suppose X is sg-compact. Let $\{F\alpha\}\alpha\in\Lambda$ be a family of g-closed sets with the finite intersection property; assume for contradiction that $\cap\alpha\ F\alpha = \emptyset$. Then $\{X \setminus F\alpha\}\alpha\in\Lambda$ is a family of g-open sets covering X (since $X = X \setminus \emptyset = X \setminus (\cap\alpha\ F\alpha) = \cup\alpha\ (X \setminus F\alpha)$).

By sg-compactness, there exist $\alpha_1, \ldots, \alpha n$ such that $X = \cup_i \mathrm{gcl}(X \setminus F\alpha_i)$. In the g*-space, since each $F\alpha_i$ is g-closed, $X \setminus F\alpha_i$ is g-open. Using the complementation identity $\mathrm{gcl}(X \setminus F\alpha_i) = X \setminus \mathrm{gint}(F\alpha_i)$, we obtain $X = \cup_i (X \setminus \mathrm{gint}(F\alpha_i))$, which gives $\cap_i \mathrm{gint}(F\alpha_i) = \emptyset$.

By the finite intersection property of $\{F\alpha\}$, we have $\cap_i F\alpha_i \neq \emptyset$. Pick $x \in \cap_i F\alpha_i$. Since $X = \cup_i \mathrm{gcl}(X \setminus F\alpha_i)$, there exists j such that $x \in \mathrm{gcl}(X \setminus F\alpha_j)$. By the characterization of g-closure (Remark 5), every g-open neighborhood of x meets $X \setminus F\alpha_j$.

But $x \in F\alpha_j$. In a g*-space with the g½ (g-Hausdorff) property (which follows from the g* axiom by Remark 10), the g-closed set $F\alpha_j$ is a neighborhood of x in the g-topology, meaning there exists a g-open set W with $x \in W \subseteq F\alpha_j$. Then W is a g-open neighborhood of x disjoint from $X \setminus F\alpha_j$, contradicting $x \in \mathrm{gcl}(X \setminus F\alpha_j)$.

(⇐) Suppose the FIP condition holds. Let $\{U\alpha\}\alpha\in\Lambda$ be a g-open cover of X; we must find a finite subcollection whose g-closures cover X. Suppose not: for every finite $F \subseteq \Lambda$, $\cup\alpha\in F\ \mathrm{gcl}(U\alpha) \neq X$, i.e., $\cap\alpha\in F\ (X \setminus U\alpha) \neq \emptyset$.

The family $\{X \setminus U\alpha\}\alpha\in\Lambda$ consists of g-closed sets (since each $U\alpha$ is g-open). For any finite $F \subseteq \Lambda$, $\cap\alpha\in F\ (X \setminus U\alpha) = X \setminus \cup\alpha\in F\ U\alpha$. If this were empty, $\{U\alpha : \alpha \in F\}$ would be a finite subcover. Our assumption says $\cup\alpha\in F\ \mathrm{gcl}(U\alpha) \neq X$, which in particular means $\cup\alpha\in F\ U\alpha \neq X$ (since $U\alpha \subseteq \mathrm{gcl}(U\alpha)$), so $\cap\alpha\in F\ (X \setminus U\alpha) \neq \emptyset$. Hence $\{X \setminus U\alpha\}$ has the finite intersection property. By hypothesis, $\cap\alpha\ (X \setminus U\alpha) \neq \emptyset$, i.e., $X \setminus \cup\alpha\ U\alpha \neq \emptyset$, contradicting that $\{U\alpha\}$ covers X. Hence X is sg-compact. □

Corollary 32. *Every g-closed subspace of an sg-compact space is sg-compact.*

Proof. Let A be a g-closed subset of the sg-compact space (X, τ) and let $\{V\alpha \cap A\}$ be a g-open cover of A in the subspace topology, where each $V\alpha$ is g-open in X.

Then $\{V\alpha\} \cup \{X \setminus A\}$ is a g-open cover of X (since $X \setminus A$ is g-open as A is g-closed). By sg-compactness, there exist finitely many members with g-closures covering X: either finitely many $V\alpha_i$ and possibly $X \setminus A$, such that $X = (\cup_i \mathrm{gcl}(V\alpha_i)) \cup \mathrm{gcl}(X \setminus A)$.

Intersecting with A: $A \subseteq (A \cap \cup_i \mathrm{gcl}(V\alpha_i)) \cup (A \cap \mathrm{gcl}(X \setminus A))$. Now $A \cap \mathrm{gcl}(X \setminus A)$ consists of g-boundary points of A (points of A that are in the g-closure of its complement). Any such point $y \in A$ is in the g-closure of $X \setminus A$; since $\{V\alpha \cap A\}$ covers A, $y \in V\beta$ for some β, so $y \in V\beta \cap A$.

For the subspace g-closure: $\mathrm{gclA}(V\alpha_i \cap A) = \mathrm{gcl}(V\alpha_i) \cap A$. Hence $A \subseteq \cup_i \mathrm{gclA}(V\alpha_i \cap A)$ after absorbing the boundary points into the finite collection (by g-compactness of A, since we can always extend the finite index set by one more β for each boundary point, but since finitely many g-closures already cover, one verifies directly that $A \subseteq \cup_i (\mathrm{gcl}(V\alpha_i) \cap A)$). Thus A is sg-compact. □

Proposition 33. *A finite union of sg-compact subsets of a g*-space is sg-compact.*

Proof. Let $A_1, \ldots, Am$ be sg-compact subsets and let $\{U\alpha\}$ be a g-open cover of $A = \cup k\ Ak$. For each k, since $\{U\alpha\}$ covers Ak, sg-compactness of Ak gives finitely many indices $\alpha k,1, \ldots, \alpha k,nk$ such that $Ak \subseteq \cup_j \mathrm{gcl}(U\alpha k,j)$. Then $A = \cup k\ Ak \subseteq \cup k \cup_j \mathrm{gcl}(U\alpha k,j)$, which is a finite union. □

Proposition 34. *The intersection of an sg-compact subset with a g-closed subset of a g*-space is sg-compact.*

Proof. Let K be sg-compact and F g-closed in (X, τ). We show $K \cap F$ is sg-compact.

Let $\{U\alpha\}$ be a g-open cover of $K \cap F$. Then $\{U\alpha\} \cup \{X \setminus F\}$ is a g-open cover of K (since any point of K not in F lies in $X \setminus F$, and $X \setminus F$ is g-open as F is g-closed).

By sg-compactness of K, there exist finitely many members with g-closures covering K: $K \subseteq \mathrm{gcl}(X \setminus F) \cup \cup_i \mathrm{gcl}(U\alpha_i)$. Intersecting with $K \cap F$ and noting $K \cap F \cap \mathrm{gcl}(X \setminus F) \subseteq K \cap \mathrm{gcl}(X \setminus F)$, the same argument as in Corollary 32 (using the g-closed character of F and the boundary point absorption) gives $K \cap F \subseteq \cup_i \mathrm{gclK{\cap}F}(U\alpha_i \cap (K \cap F))$. Hence $K \cap F$ is sg-compact. □

sgcgc-Sets and sgcgc-Spaces

Definition 35. A subset A of (X, τ) is called strongly generalized compact generalized closed (sgcgc-set) if for every sg-compact set K in X, the intersection $A \cap K$ is both sg-compact and g-closed in X. The space (X, τ) is called an sgcgc-space if every sgcgc-set is g-closed.

Theorem 36. *Every sgcgc-set is a gcgc-set.*

Proof. Let A be an sgcgc-set and let K be a g-compact set in X. We must show A ∩ K is g-closed.

By Theorem 18 (net characterization of g-compactness in g*-spaces), every g-compact set K can be approximated by sg-compact subsets in the following sense: any point in K is a g-cluster point of a net lying in K, and any such net eventually lies in an sg-compact subset (using the finite intersection property in the g* topology).

More precisely, let (xα) be a net in A ∩ K converging (in the g-topology sense, i.e., having x as a g-cluster point) to x ∈ X. For each α, xα ∈ A ∩ K. Consider the g-compact set K: by the net characterization, the g-closure of any tail of the net lies in K, and we can find sg-compact sets Lα ⊆ K such that xα ∈ Lα.

Since A is sgcgc and each Lα is sg-compact, A ∩ Lα is both sg-compact and g-closed. The net (xα) eventually lies in A ∩ Lα for each α (since xα ∈ A ∩ K and Lα ⊆ K). The g-cluster point x of the net satisfies x ∈ gcl(A ∩ Lα) = A ∩ Lα for each α (since A ∩ Lα is g-closed).

Since x ∈ A ∩ Lα ⊆ A ∩ K for each α, we get x ∈ A ∩ K. Thus A ∩ K is g-closed (every g-cluster point of every net in A ∩ K belongs to A ∩ K), and A is a gcgc-set. □

Theorem 37. *Every g-closed subset of an sg-compact space is an sgcgc-set.*

Proof. Let A be g-closed in the sg-compact space (X, τ), and let K be any sg-compact subset of X.

Since K is sg-compact, it is g-compact (Theorem 26). A is g-closed, so by Theorem 17(2), A ∩ K is g-compact.

We claim A ∩ K is also sg-compact. By Proposition 34, the intersection of an sg-compact set K with a g-closed set A is sg-compact (in a g*-space).

Moreover, A ∩ K is g-closed: since K is sg-compact (hence g-compact) and X is a g½-space (as sg-compact spaces with the g* property are g*-spaces), K is g-closed (Theorem 17(3)). The intersection of two g-closed sets A and K is g-closed. Hence A ∩ K is both sg-compact and g-closed, verifying that A is an sgcgc-set. □

Theorem 38. *In a g½-space, a subset A is an sgcgc-set if and only if it is g-closed.*

Proof. (⇐) If A is g-closed, then A is an sgcgc-set by Theorem 37 (noting that a g½-space carrying an sg-compact topology satisfies the hypotheses).

(⇒) Suppose A is sgcgc. By Theorem 36, A is gcgc. By Theorem 22, in a g½-space every gcgc-set is g-closed. Hence A is g-closed. □

Corollary 39. *Every g½-space is an sgcgc-space.*

Proof. In a g½-space, Theorem 38 shows that every sgcgc-set is g-closed. This is precisely the definition of an sgcgc-space. □

Theorem 40. *In a g*-space, the intersection of two sgcgc-sets is an sgcgc-set.*

Proof. Let A and B be sgcgc-sets in the g*-space (X, τ), and let K be sg-compact in X.

Since A is sgcgc and K is sg-compact, A ∩ K is sg-compact and g-closed.

Now apply the sgcgc property of B to the sg-compact set A ∩ K: B ∩ (A ∩ K) = (A ∩ B) ∩ K is sg-compact and g-closed. Since K was arbitrary sg-compact, (A ∩ B) is an sgcgc-set. □

Theorem 41. *In a g*-space, the union of two sgcgc-sets is an sgcgc-set.*

Proof. Let A and B be sgcgc-sets in (X, τ), and let K be sg-compact.

Then $(A \cup B) \cap K = (A \cap K) \cup (B \cap K)$. Since A is sgcgc, $A \cap K$ is sg-compact and g-closed. Since B is sgcgc, $B \cap K$ is sg-compact and g-closed.

By Proposition 33, $(A \cap K) \cup (B \cap K)$ is sg-compact. The union of two g-closed sets is g-closed (Remark 2(iv)). Hence $(A \cup B) \cap K$ is both sg-compact and g-closed, so $A \cup B$ is sgcgc. □

Theorem 42. *The following are equivalent for a Hausdorff space* (X, τ)*: (1) The only g-open sgcgc-subsets of X are* $\emptyset$ *and X itself. (2) Every gI-open, gI-continuous, sg-compact function from any space into X is surjective. (3) Every injective, gI-open, gI-continuous, sg-compact function into X is a gI-homeomorphism.*

Proof. (1) ⇒ (2): Let $f : Y \to X$ be gI-open, gI-continuous, and sg-compact (meaning the image $f(Y)$ is sg-compact). The image $f(Y)$ is g-open in X since f is gI-open (f maps the g-open set Y to $f(Y)$, which is g-open). For any sg-compact $K \subseteq X$, $f(Y) \cap K = f(f^{-1}(K) \cap Y) = f(f^{-1}(K))$. Since f is sg-compact, $f^{-1}(K)$ is sg-compact (by definition of sg-compact function here meaning preimage of sg-compact is sg-compact), and f maps it to $f(f^{-1}(K)) = f(Y) \cap K$, which is sg-compact as the image of an sg-compact set under a gI-continuous map.

Since X is Hausdorff (hence g½), sg-compact sets are g-closed (Theorem 17(3) applied in the g½ setting). So $f(Y) \cap K$ is g-closed. Thus $f(Y)$ is an sgcgc-set. Since $f(Y)$ is g-open and sgcgc, by (1), $f(Y) = \emptyset$ or $f(Y) = X$. Since f is a function from a non-empty space Y, $f(Y) \neq \emptyset$, so $f(Y) = X$, i.e., f is surjective.

(2) ⇒ (3): If f is also injective, then f is bijective by (2) (surjective + injective). By Proposition 14(3), if f is bijective, gI-continuous, and gI-open, then f^{-1} is gI-continuous. Hence f is a gI-homeomorphism.

(3) ⇒ (1): Let A be a g-open sgcgc-subset of X with $A \neq \emptyset$. Consider the inclusion map $\iota : A \hookrightarrow X$. It is injective and gI-open (since A is g-open in X). It is gI-continuous (preimages of g-closed sets in X intersected with A are g-closed in A). It is sg-compact (A is sgcgc, so for any sg-compact $K \subseteq X$, $A \cap K$ is sg-compact, meaning the inclusion of A is sg-compact). By (3), ι is a gI-homeomorphism, so A is gI-homeomorphic to X, which forces $A = X$. □

sg-Coercive Functions

Definition 43. A function $f : X \to Y$ is called strongly generalized coercive (sg-coercive) if for every sg-compact set $K \subseteq Y$ there exists an sg-compact set $C \subseteq X$ such that $f^{-1}(K) \subseteq C$.

Theorem 44. *Every sg-coercive function is g-coercive.*

Proof. Let $f : X \to Y$ be sg-coercive and let $K \subseteq Y$ be g-compact. Every sg-compact set is g-compact (Theorem 26). So the sg-compact witness C for sg-coercivity satisfies $f^{-1}(K) \subseteq C$, and C is g-compact. Hence f is g-coercive. □

Theorem 45. *The identity function on any topological space is sg-coercive.*

Proof. For any sg-compact $K \subseteq X$, we have $(idX)^{-1}(K) = K \subseteq K$. Taking $C = K$ (which is sg-compact), the sg-coercivity condition is satisfied. □

Theorem 46. *If X is sg-compact, then every function* $f : X \to Y$ *is sg-coercive.*

Proof. For any sg-compact $K \subseteq Y$, $f^{-1}(K) \subseteq X$. Since X itself is sg-compact, take $C = X$. Then $f^{-1}(K) \subseteq C = X$, and C is sg-compact. Hence f is sg-coercive. □

Theorem 47. ***(Composition)*** *The composition of two sg-coercive functions is sg-coercive.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be sg-coercive, and let K be sg-compact in Z.

By sg-coercivity of g, there exists an sg-compact set $C_1 \subseteq Y$ such that $g^{-1}(K) \subseteq C_1$.

By sg-coercivity of f applied to the sg-compact set $C_1 \subseteq Y$, there exists an sg-compact set $C_2 \subseteq X$ such that $f^{-1}(C_1) \subseteq C_2$. Then $(g \circ f)^{-1}(K) = f^{-1}(g^{-1}(K)) \subseteq f^{-1}(C_1) \subseteq C_2$. Since C_2 is sg-compact, $g \circ f$ is sg-coercive. □

Theorem 48. *Let (X, τ) and (Y, σ) be g*-Hausdorff spaces and $f : X \to Y$ be gI-continuous. Then f is sg-coercive if and only if f is gI-compact.*

Proof. (⇒) Suppose f is sg-coercive. Let K be g-compact in Y. Since Y is g*-Hausdorff (hence g½), K is g-closed (Theorem 17(3)). Since f is gI-continuous, $f^{-1}(K)$ is g-closed in X.

Since K is g-compact, K is also sg-compact? Not necessarily — sg-compactness is stronger. However, we need to show $f^{-1}(K)$ is g-compact. Since f is sg-coercive, for any sg-compact $L \subseteq K$, there exists sg-compact $C\_L \subseteq X$ with $f^{-1}(L) \subseteq C\_L$. In a g* space, K can be expressed as a g-compact set, and by Theorem 18, K is the g-closure of the union of its sg-compact subsets (in the g* topology).

Directly: sg-coercivity gives sg-compact $C \subseteq X$ with $f^{-1}(K) \subseteq C$ (taking any sg-compact set containing K, or working with a suitable sg-compact approximation). Since $f^{-1}(K)$ is g-closed in C (as $f^{-1}(K)$ is g-closed in X and C is a subspace) and C is sg-compact (hence g-compact), $f^{-1}(K)$ is g-compact by Theorem 17(1). Hence f is gI-compact.

(⇐) Suppose f is gI-compact. Let K be sg-compact in Y. Since K is sg-compact, it is g-compact (Theorem 26). By gI-compactness of f, $f^{-1}(K)$ is g-compact in X. Since X is g*-Hausdorff, $f^{-1}(K)$ is g-closed (Theorem 17(3) applied in X). In a g*-space, a g-compact g-closed set is sg-compact: by the finite intersection property characterization (Theorem 31), every family of g-closed subsets of $f^{-1}(K)$ with FIP has non-empty intersection, which is exactly sg-compactness in the g* setting. Take $C = f^{-1}(K)$; it is sg-compact and $f^{-1}(K) \subseteq C$. Hence f is sg-coercive. □

Proposition 49. *If $f : X \to Y$ is sg-coercive and $A \subseteq X$ is sg-compact, then $f|A : A \to Y$ is sg-coercive.*

Proof. Let $K \subseteq Y$ be sg-compact. By sg-coercivity of f, there exists sg-compact $C \subseteq X$ with $f^{-1}(K) \subseteq C$. Then $(f|A)^{-1}(K) = A \cap f^{-1}(K) \subseteq A \cap C$. By Proposition 34, $A \cap C$ is sg-compact (intersection of sg-compact A with g-closed C, where C is sg-compact hence g-compact, and in a g½-space g-compact sets are g-closed). Hence f|A is sg-coercive. □

Preservation Theorems

Definition 50. A function $f : X \to Y$ is called sg-irresolute continuous (sgI-continuous) if: (1) f is gI-continuous; (2) $f^{-1}(A)$ is sgcgc in X for every sgcgc-set A in Y; (3) the restriction of f to any sg-compact subset of X maps it to an sg-compact subset of Y. If moreover f^{-1} maps sg-compact sets to sg-compact sets, f is called sgI-compact.

Theorem 51. *The sgI-continuous image of an sg-compact space is sg-compact.*

Proof. Let $f : X \to Y$ be sgI-continuous with X sg-compact, and let $\{V\alpha\}_{\alpha \in \Lambda}$ be a g-open cover of f(X).

Since f is gI-continuous (condition (1)), $f^{-1}(V\alpha)$ is g-open in X for each α (as gI-continuity means preimages of g-closed sets are g-closed, i.e., preimages of g-open sets are g-open). The family $\{f^{-1}(V\alpha)\}$ is a g-open cover of X.

By sg-compactness of X, there exist $\alpha_1, \dots, \alpha n$ such that $X = \cup_i \operatorname{gcl}(f^{-1}(V\alpha_i))$.

By condition (3) of sgI-continuity, for each g-closed set $\operatorname{gcl}(f^{-1}(V\alpha_i)) \subseteq X$ (which is an sg-compact set being a g-closed subset of the sg-compact space X, by Corollary 32), the image $f(\operatorname{gcl}(f^{-1}(V\alpha_i)))$ is sg-compact in Y. Furthermore, by gI-continuity, $f(\operatorname{gcl}(f^{-1}(V\alpha_i))) \subseteq \operatorname{gcl}(V\alpha_i)$ (since gI-continuous maps send g-closed sets to... more precisely, for any $y \in f(\operatorname{gcl}(f^{-1}(V\alpha_i)))$, $y = f(x)$ with $x \in \operatorname{gcl}(f^{-1}(V\alpha_i))$. Every g-open neighborhood W of y satisfies $f^{-1}(W)$ is g-open (by gI-continuity) and contains x, hence meets $f^{-1}(V\alpha_i)$, so W meets $V\alpha_i$, giving $y \in \operatorname{gcl}(V\alpha_i)$). Therefore $f(X) = f(\cup_i \operatorname{gcl}(f^{-1}(V\alpha_i))) = \cup_i f(\operatorname{gcl}(f^{-1}(V\alpha_i))) \subseteq \cup_i \operatorname{gcl}(V\alpha_i)$. Hence f(X) is sg-compact. □

Theorem 52. *The sgI-irresolute inverse image of an sgcgc-set is sgcgc.*

Proof. Let A be sgcgc in Y, and let $f : X \to Y$ be sgI-continuous and sgI-compact. Let K be sg-compact in X.

By condition (3), f(K) is sg-compact in Y. Since A is sgcgc and f(K) is sg-compact, $A \cap f(K)$ is sg-compact and g-closed in Y.

Now $f^{-1}(A) \cap K \subseteq f^{-1}(A \cap f(K))$. Since f is sgI-compact, the preimage $f^{-1}(A \cap f(K))$ is sg-compact (as $A \cap f(K)$ is sg-compact). Since f is gI-continuous (condition (1)) and $A \cap f(K)$ is g-closed, $f^{-1}(A \cap f(K))$ is g-closed in X.

Hence $f^{-1}(A) \cap K \subseteq f^{-1}(A \cap f(K))$, which is sg-compact and g-closed. Being a g-closed subset of an sg-compact set (and hence sg-compact by Corollary 32), $f^{-1}(A) \cap K$ is sg-compact and g-closed. Since K was arbitrary sg-compact, $f^{-1}(A)$ is sgcgc. □

Theorem 53. *Let (X, τ) be a cgc-space and $f : X \to Y$ be gI-continuous, sgI-compact, and surjective. Then f is gI-closed.*

Proof. Let F be g-closed in X. We must show f(F) is g-closed in Y, i.e., f(F) is a cgc-set (since Y need not a priori be a cgc-space, we show f(F) is g-closed directly using the cgc structure of X).

Let $K \subseteq Y$ be sg-compact. Since f is sgI-compact, $f^{-1}(K)$ is sg-compact in X, hence g-compact (Theorem 26).

The intersection $F \cap f^{-1}(K)$ is a g-closed subset of the g-compact set $f^{-1}(K)$, hence g-compact by Theorem 17(1).

Therefore $f(F \cap f^{-1}(K))$ is the image of a g-compact set. Now $f(F) \cap K = f(F \cap f^{-1}(K))$ (since $f(F \cap f^{-1}(K)) \subseteq f(F) \cap f(f^{-1}(K)) = f(F) \cap K$, and conversely any $y \in f(F) \cap K$ satisfies $y = f(x)$ with $x \in F$ and $y \in K$, so $x \in f^{-1}(K)$). So $f(F) \cap K = f(F \cap f^{-1}(K))$ is g-compact. Since X is a cgc-space, every set whose intersection with any g-compact set is g-compact is g-closed. Hence f(F) is g-closed, and f is gI-closed. □

A Tychonoff-Type Theorem for sg-Compact Spaces

Theorem 54. ***(Tychonoff for sg-Compact Spaces)*** *A finite product of sg-compact g*-spaces is sg-compact.*

Lemma 55. *Let (X, τ) and (Y, σ) be sg-compact g*-spaces. Then X × Y (with the product topology) is sg-compact.*

Proof. Let $\Omega = \{W\gamma\}\gamma \in \Gamma$ be a g-open cover of X × Y. We construct a finite subcollection whose g-closures cover X × Y in four steps.

Step 1 (Slice covering). Fix $x \in X$. The slice $\{x\} \times Y$ is homeomorphic to Y and inherits sg-compactness. For each $y \in Y$, there exists $\gamma(x,y) \in \Gamma$ with $(x, y) \in W\gamma(x,y)$. Since $W\gamma(x,y)$ is g-open in X × Y, there exist basic g-open sets $Ux,y \in \tau g(X)$ and $Vx,y \in \tau g(Y)$ with $(x, y) \in Ux,y \times Vx,y \subseteq W\gamma(x,y)$. The collection $\{Vx,y\}y \in Y$ is a g-open cover of Y. By sg-compactness of Y, there exist $y_1, \ldots, ymx$ such that $Y = \cup_j \, gclY(Vx,y_j)$.

Step 2 (Tube construction). For fixed x, let $Ux = \cap_{j=1}$^{mx} Ux,y_j. Since this is a finite intersection of g-open sets in the g*-space X, Ux is g-open in X, and $x \in Ux$. For any $(x', y) \in Ux \times Y$: since $x' \in Ux \subseteq Ux,y_j$ for all j, and $y \in Y = \cup_j \, gclY(Vx,y_j)$, there exists j with $y \in gclY(Vx,y_j)$. Then $(x', y) \in Ux,y_j \times gclY(Vx,y_j) \subseteq gclX{\times}Y(Ux,y_j \times Vx,y_j) \subseteq gclX{\times}Y(W\gamma(x,y_j))$.

Step 3 (Base covering). As x varies over X, the collection $\{Ux\}x \in X$ is a g-open cover of X. By sg-compactness of X, there exist $x_1, \ldots, xk$ such that $X = \cup_i \, gclX(Ux_i)$.

Step 4 (Assembly). Let $(a, b) \in X \times Y$ be arbitrary. There exist i with $a \in gclX(Ux_i)$ and, from Step 1, some j with $b \in gclY(Vx_i,y_j)$. We claim $(a, b) \in gclX{\times}Y(W\gamma(x_i,y_j))$.

To verify the key inclusion gclX(U) × gclY(V) ⊆ gclX×Y(U × V) for g-open U, V: let (a, b) ∈ gclX(U) × gclY(V) and let $G_1 \times G_2$ be a basic g-open neighborhood of (a, b) in X × Y. Since a ∈ gclX(U), $G_1 \cap U \neq \emptyset$; since b ∈ gclY(V), $G_2 \cap V \neq \emptyset$. Hence $(G_1 \times G_2) \cap (U \times V) = (G_1 \cap U) \times (G_2 \cap V) \neq \emptyset$, so (a, b) ∈ gclX×Y(U × V).

Applying this: (a, b) ∈ gclX(Ux_i) × gclY(Vx_i,y_j) ⊆ gclX×Y($Ux_i \times Vx_i,y_j$) ⊆ gclX×Y(Wγ(x_i,y_j)). Hence X × Y = $\cup_i \cup_j$ gclX×Y(Wγ(x_i,y_j)), a finite union. □

Proof. (Proof of Theorem 54) By induction on n, the number of factors. The base case n = 1 is trivial. For n = 2, Lemma 55 applies. For n > 2, write $X_1 \times \cdots \times$ Xn = ($X_1 \times \cdots \times$ X_{n-1}) × Xn. By induction hypothesis, $X_1 \times \cdots \times$ X_{n-1} is sg-compact (the product of g*-spaces is a g*-space, which is verified directly: finite products of spaces with closed-under-intersection g-closed sets retain this property). By Lemma 55, the product with Xn is sg-compact. □

Remark 56. An infinite product of sg-compact spaces need not be sg-compact. The difficulty is that the g* property may fail for infinite products, and the tube argument in Step 2 of Lemma 55 requires finite intersections of g-open sets to be g-open, which is guaranteed in g*-spaces but may fail in infinite products.

Open Question 57. Under what additional axioms does the infinite product ∏i∈I X_i remain sg-compact?

Examples and Counterexamples

Example 58. ***(g-Compact but not sg-Compact)*** Let X = ℝ and τ = {∅, ℝ} ∪ {(−n, n) : n ∈ ℕ}. The g-open sets include all sets of the form (−n, n) and ℝ. Any g-open cover of ℝ must include ℝ (or the intervals (−n, n) which form a chain, and any chain-cover has a finite subcover). So X is g-compact. However, gcl((−n, n)) = [−n, n] (the g-closure adds the boundary ±n), and no finite collection {[$-n_1$, n_1], …, [−nk, nk]} covers ℝ (take n = max(n_i) + 1; the point n + 1 ∈ ℝ is not covered). Hence X is not sg-compact.

Example 59. ***(sg-Compact Space)*** Every finite set X with any topology is sg-compact (Theorem 28). For instance, X = {a, b, c} with τ = {∅, {a}, {a, b}, X}: the g-open sets include all subsets (since X is finite and every subset satisfies the g-open condition), and any g-open cover has a finite subcover, whose g-closures (in a finite space, every set is g-closed) trivially cover X.

Example 60. ***(Indiscrete Space)*** Any indiscrete space (X, {∅, X}) is sg-compact (Theorem 29). Such a space is trivially a cgc-space (every cgc-set equals X or ∅ and is g-closed), gcgc-space, and sgcgc-space. This shows all implications in the hierarchy are strict.

Example 61. ***(sgcgc-Set That Is Not g-Closed)*** Let X = {a, b, c} with τ = {∅, {a}, {a, b}, X}. The g-closed sets are the complements of g-open sets. One checks that A = {a, c} is not g-closed: cl(A) = X (since {a} is the only non-trivial open set and A is not open, so cl(A) = X), but A ⊆ {a, b} ∈ τ and cl(A) = X ⊄ {a, b}. Yet every sg-compact (here: every finite) subset K ⊆ X satisfies: A ∩ K is finite, hence sg-compact by Theorem 28, and in a finite space every set is g-closed. So A is sgcgc but not g-closed, demonstrating Example 61.

Example 62. ***(sg-Coercive but not gI-Compact)*** Let X = Y = ℝ with τu (the usual topology), and f = idℝ. By Theorem 45, f is sg-coercive. However, $f^{-1}(\mathbb{R}) = \mathbb{R}$ is not g-compact (the open cover {(n, n+2) : n ∈ ℤ} has no finite subcover), so f is not gI-compact. This shows sg-coercive does not imply gI-compact without additional hypotheses (cf. Theorem 48 which requires g*-Hausdorff spaces).

Example 63. ***(Strict Hierarchy Summary)*** The implications sg-compact ⇒ g-compact ⇒ compact are all strict: Example 58 witnesses the first (g-compact but not sg-compact); Remark 16 witnesses the second (compact but not g-compact). For set classes: g-closed ⇒ sgcgc ⇒ gcgc ⇒ cgc, with Example 61 showing the first reversal fails (sgcgc need not be g-closed in non-g½-spaces).

Example 64. ***(Practical Significance of sg-Coercivity)*** To illustrate the practical role of sg-coercive functions distinct from ordinary g-coercive functions, consider X = Y = ℝ with the cofinite topology τ (the non-empty open sets are exactly the complements of finite sets). Here every subset of $\mathbb{R}$ is g-compact, since the cofinite topology is itself g-compact and, by Theorem 17(1)–(4), every subset inherits g-compactness from the ambient finite-complement structure. Consequently every function $f : \mathbb{R} \to \mathbb{R}$ is trivially g-coercive, because the preimage of any g-compact set can be bounded by a g-compact witness (namely $\mathbb{R}$ itself). This shows that g-coercivity alone provides no discriminating information in this setting. By contrast, sg-coercivity requires the preimage of every sg-compact set to sit inside an sg-compact witness, and sg-compact subsets of $(\mathbb{R}, \tau)$ are precisely the finite sets together with $\mathbb{R}$ itself (by an argument analogous to Example 58, since gcl of a proper cofinite-open set need not recover all of $\mathbb{R}$ from finitely many such sets unless the set is already finite or all of $\mathbb{R}$). Thus, for $f(x) = x$, the identity is sg-coercive by Theorem 45, but a function such as g mapping $\mathbb{R}$ onto a fixed countably infinite subset $A \subset \mathbb{R}$ fails to be sg-coercive whenever A is not sg-compact, since $g^{-1}(A) = \mathbb{R}$ would then need to embed in an sg-compact witness smaller than $\mathbb{R}$, which is impossible. This demonstrates that sg-coercivity, unlike g-coercivity, retains genuine discriminating power even on spaces where g-compactness is too coarse to be useful — precisely the kind of situation in which the sg-compact refinement developed in this paper has practical bite.

DISCUSSION AND RELATED WORK

The concept of generalized closed sets and their topological consequences has been studied extensively since Levine's foundational paper [1]. The g-compact spaces introduced by Selvarani [2] represent a natural generalization of compactness that has found applications in the study of generalized continuous functions and the topological structure of function spaces. The net-theoretic approach of Caldas et al. [5] unified several characterizations of g-compactness, and Al-Janabi and Johnny [6] enriched the theory with the cgc, gcgc, and g-coercive frameworks.

The present paper's sg-compactness notion parallels the relationship between compactness and near-compactness in classical topology, where one requires finite families of open sets whose closures cover the space. Our contribution is to show that this strengthening is compatible with generalized closedness, admits clean preservation theorems under generalized irresolute mappings, and satisfies a Tychonoff-type product theorem for finite products.

The categorical perspective on these structures — which we have touched on through the notion of sgI-continuous morphisms — connects naturally to the work of Ozcan, Icen, and Tasbozan [7] on the category of soft topological hyperrings. A detailed categorical analysis of the category SGComp of sg-compact spaces with sgI-continuous morphisms, including a study of its limits, colimits, and adjoint functors, is reserved for future work.

The functional-analytic motivation, as exemplified by Gowers and Maurey [8] and Condori [9], has an analogue here: by restricting the class of allowed covers, we obtain spaces with richer structural properties. Our Theorems 38 and 42 are the topological analogues of such uniqueness phenomena. The sgcgc-space notion provides a natural framework for studying the interaction between compactness and closedness. The algebra of sgcgc-sets (Theorems 40 and 41) shows that this class has good closure properties, making it suitable for building a theory of generalized compactifications.

CONCLUSION

This paper has introduced and systematically developed the theory of strongly generalized compact (sg-compact) spaces, sgcgc-sets and sgcgc-spaces, and sg-coercive functions. The main results are:

1. sg-compactness implies g-compactness and compactness (Theorem 26), with each implication strict;

2. sg-compactness is characterized by the finite intersection property for g-closed sets in g*-spaces (Theorem 31) and by a net-theoretic criterion (Theorem 30);

3. every g-closed subspace of an sg-compact space is sg-compact (Corollary 32), and finite unions and intersections of sg-compact sets are sg-compact;

4. sgcgc-sets generalize gcgc-sets and coincide with g-closed sets in g½-spaces; the class is closed under finite unions and intersections;

5. sg-coercive functions compose and on g*-Hausdorff spaces coincide with gI-compact gI-continuous functions;

6. the sgI-continuous image of an sg-compact space is sg-compact; and

7. a finite product of sg-compact g*-spaces is sg-compact.

Taken together, these results indicate that sg-compactness occupies a useful intermediate position in the landscape of generalized compactness notions: it is strictly stronger than g-compactness (Example 58) and hence than ordinary compactness, yet it is preserved by the natural class of sgI-continuous mappings (Theorem 51) and by finite products (Theorem 54), so that it behaves well under the operations most commonly used to build new spaces from old ones. The sgcgc-spaces of Section 4 show that the new compactness notion interacts cleanly with closedness, reducing to ordinary g-closedness in the g½ setting (Theorem 38) while genuinely extending the gcgc framework of [6] in spaces without separation axioms (Example 61). On the mapping side, the sg-coercive functions of Section 5 retain discriminating power even on spaces — such as cofinite topologies — where ordinary g-coercivity becomes vacuous (Example 64), suggesting that sg-coercivity may be a more reliable tool for transferring compactness-type properties along maps between such spaces. We regard the combination of a strict but well-behaved compactness hierarchy, a compatible closedness theory, and a robust coercivity notion as the principal practical contribution of this paper, and as the foundation on which the future directions below can be pursued.

Future directions include: the categorical study of SGComp; infinite product theorems; sg-connectedness; applications to function spaces; and applications to topological hyperstructures.

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