Exponentiated Generalized Modified Weibull Distribution for Skewed Dataset

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ABSTRACT

Using the Exponentiated approach and three-parameter Weibull distribution as baseline function, a newly generalized distribution was formed called the Exponentiated Generalized Modified Weibull distribution. One of the properties of a proper probability density function was used to ascertain that the resulting function is a proper probability density function. Statistical properties of the newly generated distribution were studied and graphs of probability density and cumulative density functions of the distribution were plotted using varying parameter values. Monte Carlo simulation approach was used for the test of homogeneity of the distribution and it was observed that the parameters in the distribution approach the true value as sample size increases. The distribution was compared with some of the existing distributions in its category and it was observed that the distribution outperformed the existing distributions using secondary data. Therefore, it was concluded that Exponentiated Generalized Modified Weibull distribution can be adopted in modeling events involving distributions of its category.

Keywords: Probability density function, cumulative density function, survival function, hazard function and modeling

INTRODUCTION

Some of the often utilized lifespan distributions include the exponential, Weibull, gamma, Rayleigh, Pareto, and Gompertz distributions, which have monotonic hazard rate functions, according to Lawless [12]. The bathtub shape, the unimodal (upside-down bathtub shape), or the modified unimodal shape are examples of non-monotonic forms that are necessary for particular lifetime data, such as mortality, life cycles, and data from specific biological and medical investigations.

Among the most significant, coveted, and often utilized lifespan distributions is the distribution known as Weibull. It has numerous uses in various sectors, including clinical studies and engine failure. The Weibull distribution's cumulative density function has a closed form that provides a straightforward representation of its survival and hazard functions. This adaptable distribution can be used for various lifetime data sets in many fields. Its parameters are estimable and it has a physical meaning.

The two-parameter Weibull distribution has wide coverage in terms of application which gave it more recognition among rersearchers but one of the flaws it has is that it does not show any sort of non-monotonic hazard rate shape (Aldahlan and Afify, [2]). For many years, to generate non-monotonic shapes, researchers have tweaked the Weibull distribution in numerous ways using diverse methodologies.

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The two-parameter flexible Weibull extension of Bebbington *et al*., [5] has a hazard function that can be increasing, decreasing, or bathtub-shaped. Zhang and Xie [18] studied the characteristics and application of the truncated Weibull distribution which has a bathtub-shaped hazard function.

A three-parameter model, called the exponentiated Weibull distribution, was introduced by Mudholkar and Srivastava [14] and another three-parameter model was introduced by Marshall and Olkin [13]. Extended Weibull distribution was proposed by. Xie *et al.*, [17] which has three parameters, and was referred to as modified Weibull extension with a bathtub-shaped hazard function. Awopeju and Alfred [4] presented a five-parameter form of Weibull distribution called Generalized Modified Weibull distribution which was formed using the T{x} approach with the claim that the generalized distribution is capable of modeling skewed dataset. The modified Weibull (MW) distribution by Lai *et al* [11] multiplies the Weibull cumulative hazard function with a new function which was later generalized to exponentiated form by Carrasco *et al* [6]. Chen [7] also put forward a certain variant of the Weibull distribution in what is today referred to as the Chen distribution with non-monotone properties.

Since Weibull distribution is an essential part of the survival model, the modified Weibull distributions generated by researchers were expected to lead to more robust survival models in the course of analysis. In the same vein, the gamma and exponential distributions have been worked on by researchers which have resulted in various modifications of these distributions.

As part of the motivation for this research, the behaviour of a modified Weibull distribution was considered and generalized to derive a new distribution in the same category. The research is aimed at proposing a new generalized Weibull-class Survival Distribution by generalizing the Modified Weibull Distribution. Also, to establish statistical properties of the generalized distribution and estimate the parameters of the newly generalized Exponentiated Modified Weibull distribution. A simulation study was used in the study to ascertain the homogenous property of the distribution and the distribution was compared with existing distributions in its category to measure its suitability in the field of Statistics.

The study contributes to the pool of literature in the study area and stands to generate a more robust Weibull class of distribution.

MATERIAL AND METHODS

Using the **Modified Weibull Distribution by Lai *et al*. [11]** with cumulative density function and probability density function given respectively by

$$F(x) = 1 - \exp\left[-\beta x^{\gamma} e^{\lambda x}\right], \qquad (1)$$

$$f(x) = \beta(\gamma + \lambda x)x^{\gamma-1} e^{\lambda x} \exp\left[-\beta x^{\gamma} e^{\lambda x}\right], \quad (2)$$

$$x > 0, \beta > 0, \lambda > 0, \gamma > 0,$$

as baseline function, where $\beta \; and \; \lambda$ are scale parameters while γ is the shape parameter. The goal of the research is to generalize the distribution in Equations (2) using ***exponentiated generalized method [10]***. The method for generating new distribution by generalization using the cumulative density function is of the form;

$$G(x) = \left[1 - \left(1 - F(x)\right)^{k}\right]^{s}, \; k, s > 0, \qquad (3)$$

where $k \; and \; s$ are shape parameters. In Equation (3), $F(x)$ is taken to be the cumulative density function of any desired distribution. In this regard, we shall take $F(x)$ to be the Modified Weibull distribution in Equation (1). Observe that in the distribution, there are three parameters. Putting $F(x)$ in (1) into Equation (3) leads to a more flexible five-parameter distribution since the parameters $k, and \; s$ were added.

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Using F(x) of the Modified Weibull distribution, the Cumulative Density Function of the Exponentially Generated Modified Weibull Distribution (EGMW) is defined as;

$$G(x) = \left[1 - \left(1 - (1 - e^{-\beta x^r e^{\lambda x}}\right)^k\right]^s \qquad (4)$$

Simply,

$$G(x) = \left[\left(1 - e^{-k\beta x^r e^{\lambda x}}\right)\right]^s, k, \beta, r. \lambda, s, x > 0 \qquad (5)$$

Therefore, the cumulative density function of Exponentiated Generated Modified Weibull distribution is $\left[\left(1 - e^{-k\beta x^r e^{\lambda x}}\right)\right]^s$

Taking the derivative of the cumulative density function of the Exponentiated Generated Modified Weibull distribution results in a probability density function;

$$f(x) = k\beta s x^{\lambda-1}(\lambda\ + rx)e^{rx}e^{-k\beta x^{\lambda}e^{rx}}\left[1 - e^{-k\beta x^{\lambda}e^{rx}}\right]^{s-1}, \qquad (6)$$

$$x > 0, s\ > 0, k > 0, \beta\ > 0, \lambda > 0, r>0$$

Area under Curve of the newly Generated Distribution

For a proper probability density function, $\int_{-\infty}^{\infty} f(x)dx = 1$

So, let f(x) be probability density function of Exponentiated Generated Modified Weibull distribution, then;

$$\int_{\infty}^{\infty} f(x)dx = k\beta s \int_0^{\infty} x^{\lambda-1}(\lambda + rx)e^{rx}e^{-k\beta x^{\lambda}e^{rx}}\left[1 - e^{-k\beta x^{\lambda}e^{rx}}\right]^{s-1}dx$$

Let $u = e^{-k\beta x^{\lambda}e^{rx}}$ then

$$dx\ \frac{du}{-ku\beta x^{\lambda-1}(\lambda\ + rx)e^{rx}}$$

By substitution,

$$= \int_1^0 -s(1-u)^{s-1}du \quad = s\int_1^0 (1-u)^{s-1}du = 1$$

Hence, the expression is a complete probability density function.

Survival Characteristics of Exponentiated Generated Modified Weibull Distribution

The survival characteristics of the distribution can be obtained using Equations 7 and 8.. Let the probability distribution function of a distribution be f(x) and the corresponding Cumulative Density Function be F(x). The Survival function S(x) is;

$$S(x) = (1 - F(x)) \qquad (7)$$

Hazard function (h(x)) of the distribution becomes;

$$h(x) = \frac{f(x)}{S(x)} \qquad (8)$$

Therefore, the hazard function can be expressed as

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$$h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{(1-F(x))} = \frac{F^{|}(x)}{(1-F(x))} \qquad (9)$$

Using the above expression, the cumulative density function and probability density function of the Exponentiated Generated Modified Weibull distribution are

$$F(x) = \left[1 - e^{-k\beta x^{\lambda} e^{rx}}\right]^{s}, \quad x > 0, s > 0, k > 0, \beta > 0, \lambda > 0, r > 0$$

and

$$f(x) = k\beta s x^{\lambda-1}(\lambda + rx)e^{rx}e^{-k\beta x^{\lambda} e^{rx}}\left[1 - e^{-k\beta x^{\lambda} e^{rx}}\right]^{s-1}, x > 0, s > 0, k > 0, \beta > 0, \lambda > 0, r > 0.$$

The survival function becomes;

$$S(x) = 1 - \left[1 - e^{-k\beta x^{\lambda} e^{rx}}\right]^{s} \qquad (10)$$

The hazard function is;

$$h(x) = \frac{k\beta s x^{\lambda-1}(\lambda + rx)e^{rx}e^{-k\beta x^{\lambda} e^{rx}}\left[1-e^{-k\beta x^{\lambda} e^{rx}}\right]^{s-1}}{1-\left[1-e^{-k\beta x^{\lambda} e^{rx}}\right]^{s}} \qquad (11)$$

Graphs of the Exponentiated Generated Modified Weibull Distribution

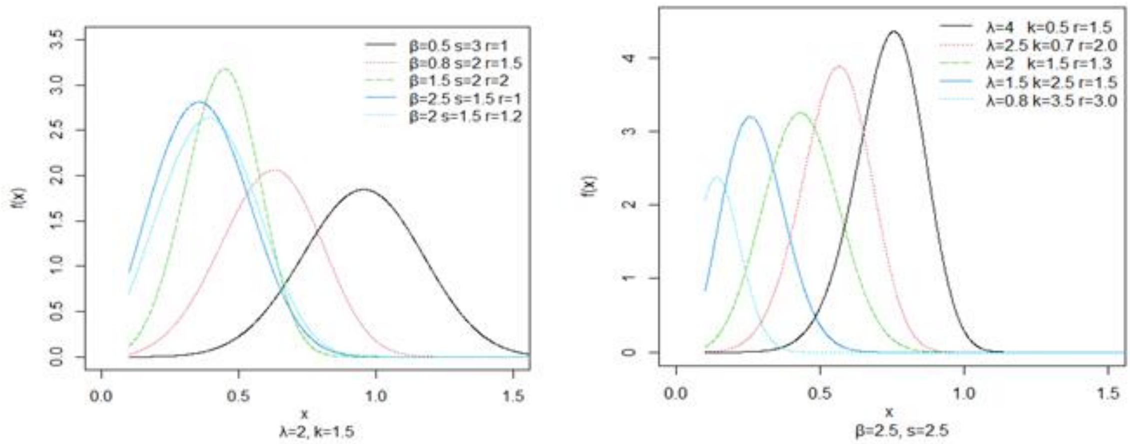

Figure 1: Probability density function of EGMW distribution for varying parameter values.

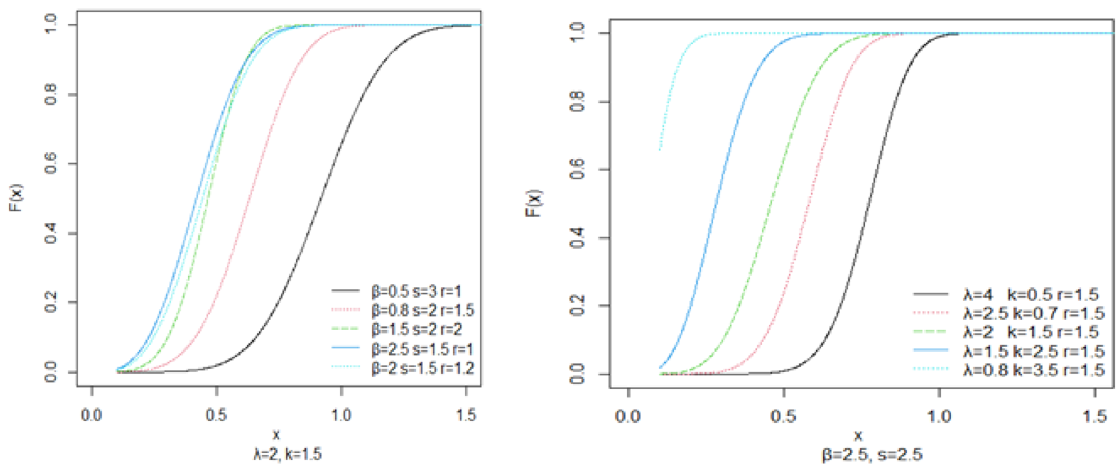

Figure 2: cumulative density function of EGMW distribution for varying parameter values.

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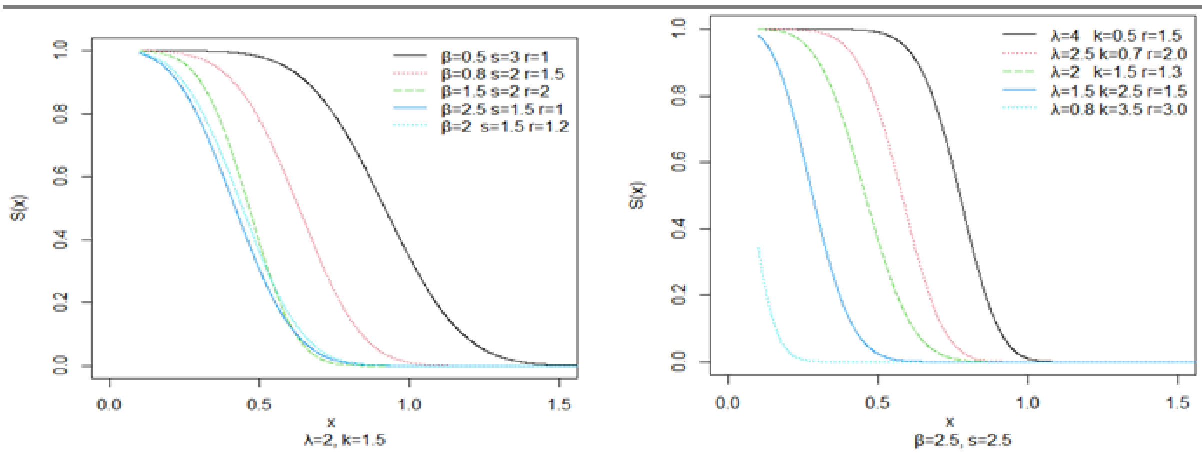

Figure 3: Survival function plot of EGMW distribution for varying parameter values.

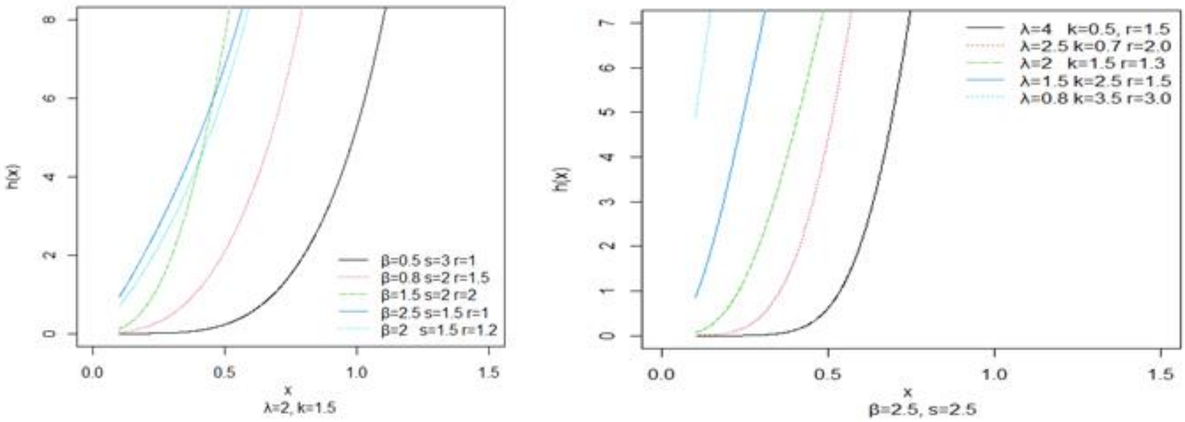

Figure 4: Hazard function plot of EGMW distribution for varying parameter values

From the graphs, it can be observed that the generalized distribution has a shape of typical probability density function as well as the cumulative density function. Flexibility of the distribution can be seen in the graph of the probability density function (pdf) of Exponentiated Generalized Modified Weibull (EGMW) Distribution. It can also be observed that the distribution is capable of modeling skewed data considering the shape of its pdf.

Properties of Exponentially Generated Modified Weibull (EGMW) Distribution

The following are the statistical properties of the newly generated distribution;

Moments

Suppose a random variable X is EGMW distributed, its p^{th} moment is given in the following theorem;

Theorem 1: *Suppose that the random variable X follows the EGMW distribution, then, its p^{th} non-central moment is given as*

$$E[X^p] = k\beta s \sum_{j=0}^{\infty} \binom{s-1}{j} (-1)^j \sum_{m=0}^{\infty} \frac{(-k\beta(1+j))^m \Gamma(p+\lambda+\lambda m)}{m!\,(-r(m+1))^{(p+\lambda+\lambda m)}} \left[\lambda - \frac{p+\lambda+\lambda m}{m+1}\right]$$

and its mean is given as

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$$E[X] = k\beta s \sum_{j=0}^{\infty} \binom{s-1}{j} (-1)^j \sum_{m=0}^{\infty} \frac{(-k\beta(1+j))^m \Gamma(1+\lambda+\lambda m)}{m!\,(-r(m+1))^{(1+\lambda+\lambda m)}} \left[\lambda - \frac{1+\lambda+\lambda m}{m+1}\right]$$

Proof

The pth non-central moment is defined by

$$E[X^p] = \int_0^{\infty} x^p f(x) dx$$

Suppose $f(x)$ is the pdf of EGMW distribution, then

$$E[X^p] = k\beta s \int_0^{\infty} x^{p+\lambda-1}(\lambda + rx)e^{rx}e^{-k\beta x^{\lambda}e^{rx}} \left[1 - e^{-k\beta x^{\lambda}e^{rx}}\right]^{s-1} dx \qquad (12)$$

Using the binomial expansion approach,

$$\left[1 - e^{-k\beta x^{\lambda}e^{rx}}\right]^{s-1} = \sum_{j=0}^{\infty} \binom{s-1}{j} (-1)^j e^{-k\beta x^{\lambda}e^{rx} j} \qquad (13)$$

Substituting Equation (13) into Equation (12), we have

$$E[X^p] = k\beta s \sum_{j=0}^{\infty} \binom{s-1}{j} (-1)^j \int_0^{\infty} x^{p+\lambda-1}(\lambda + rx)e^{rx}e^{-k\beta x^{\lambda}e^{rx}(1+j)} dx \qquad (14)$$

Apply power series, the exponential function is

$$e^{-k\beta x^{\lambda}e^{rx}(1+j)} = \sum_{m=0}^{\infty} \frac{(-k\beta(1+j))^m x^{\lambda m} e^{rmx}}{m!} \qquad (15)$$

Substituting Equation (15) into Equation (14), it results to;

$$E[X^p] = k\beta s \sum_{j=0}^{\infty} \binom{s-1}{j} (-1)^j \sum_{m=0}^{\infty} \frac{(-k\beta(1+j))^m}{m!} \int_0^{\infty} x^{p+\lambda+\lambda m-1}(\lambda + rx)e^{x(r+rm)} dx \qquad (16)$$

Consider the integrand in Equation (16);

$$\int_0^{\infty} x^{p+\lambda+\lambda m-1}(\lambda + rx)e^{x(r+rm)} dx = \frac{\Gamma(p+\lambda+\lambda m)}{(-r(m+1))^{(p+\lambda+\lambda m)}} \left[\lambda - \frac{p+\lambda+\lambda m}{m+1}\right]$$

By substitution Equation (16) becomes;

$$E[X^p] = k\beta s \sum_{j=0}^{\infty} \binom{s-1}{j} (-1)^j \sum_{m=0}^{\infty} \frac{(-k\beta(1+j))^m \Gamma(p+\lambda+\lambda m)}{m!\,(-r(m+1))^{(p+\lambda+\lambda m)}} \left[\lambda - \frac{p+\lambda+\lambda m}{m+1}\right] \qquad (17)$$

The mean is obtained by setting $p = 1$ in Equation (17);

$$E[X] = k\beta s \sum_{j=0}^{\infty} \binom{s-1}{j} (-1)^j \sum_{m=0}^{\infty} \frac{(-k\beta(1+j))^m \Gamma(1+\lambda+\lambda m)}{m!\,(-r(m+1))^{(1+\lambda+\lambda m)}} \left[\lambda - \frac{1+\lambda+\lambda m}{m+1}\right] \qquad (18)$$

The q^{th} central moment is given in the following theorem.

Theorem 2: Suppose that the random variable X follows the EGMW distribution, then its q^{th} central moment is given as

$$E[X-\mu]^q = k\beta s \sum_{p=0}^{q} \binom{q}{p} (-\mu)^{q-p} \sum_{j=0}^{\infty} \binom{s-1}{j} (-1)^j$$

$$\sum_{m=0}^{\infty} \frac{(-k\beta(1+j))^m \Gamma(p+\lambda+\lambda m)}{m!\,(-r(m+1))^{(p+\lambda+\lambda m)}} \left[\lambda - \frac{p+\lambda+\lambda m}{m+1}\right]$$

and the variance is given as

$$E[X-\mu]^2 = k\beta s \sum_{p=0}^{2} \binom{q}{p} (-\mu)^{2-p} \sum_{j=0}^{\infty} \binom{s-1}{j} (-1)^j$$

$$\sum_{m=0}^{\infty} \frac{(-k\beta(1+j))^m \Gamma(p+\lambda+\lambda m)}{m!\,(-r(m+1))^{(p+\lambda+\lambda m)}} \left[\lambda - \frac{p+\lambda+\lambda m}{m+1}\right]$$

Proof

The q^{th} central moment is defined by

$$E[X-\mu]^q = k\beta s \sum_{p=0}^{q} \binom{q}{p} (-\mu)^{q-p} E[X^p] \qquad (19)$$

Where $\mu = E[X]$ and $E[X^p]$ are given in Equation (17). By substitution,

$$E[X-\mu]^q = k\beta s \sum_{p=0}^{q} \binom{q}{p} (-\mu)^{q-p} \sum_{j=0}^{\infty} \binom{s-1}{j} (-1)^j$$

$$\sum_{m=0}^{\infty} \frac{(-k\beta(1+j))^m \Gamma(p+\lambda+\lambda m)}{m!\,(-r(m+1))^{(p+\lambda+\lambda m)}} \left[\lambda - \frac{p+\lambda+\lambda m}{m+1}\right] \quad (20)$$

which is the central moment of EGMW distribution.

Its variance is obtained by setting $q = 2$ in Equation (20) to obtain

$$E[X-\mu]^2 = k\beta s \sum_{p=0}^{2} \binom{q}{p} (-\mu)^{2-p} \sum_{j=0}^{\infty} \binom{s-1}{j} (-1)^j$$

$$\sum_{m=0}^{\infty} \frac{(-k\beta(1+j))^m \Gamma(p+\lambda+\lambda m)}{m!(-r(m+1))^{(p+\lambda+\lambda m)}} \left[\lambda - \frac{p+\lambda+\lambda m}{m+1}\right] \quad (21)$$

Moment Generating Function

The moment generating function of the EGMW is given in the following theorem.

Theorem 3: Let X follows the EGMW, and then moment generating function is

$$M_x(t) = k\beta s \sum_{p=0}^{\infty} \frac{t^p}{p!} \sum_{j=0}^{\infty} \binom{s-1}{j} (-1)^j \sum_{m=0}^{\infty} \frac{(-k\beta(1+j))^m \Gamma(p+\lambda+\lambda m)}{m!\,(-r(m+1))^{(p+\lambda+\lambda m)}} \left[\lambda - \frac{p+\lambda+\lambda m}{m+1}\right]$$

Proof

The moment generating function of a random variable X is given by

$$M_x(t) = E(e^{tx}) = \sum_{r=0}^{\infty} \frac{t^p}{p!} E[X^p]. \qquad (22)$$

If X follows EGMW distribution, then its moment is given in Equation (17) into Equation (22),

$M_x(t) =$

$k\beta s \sum_{p=0}^{\infty} \frac{t^p}{p!} \sum_{j=0}^{\infty} \binom{s-1}{j} (-1)^j \sum_{m=0}^{\infty} \frac{(-k\beta(1+j))^m \Gamma(p+\lambda+\lambda m)}{m!(-r(m+1))^{(p+\lambda+\lambda m)}} \left[\lambda - \frac{p+\lambda+\lambda m}{m+1}\right]$ (23)

The q^{th} quantile function

It is the real solution of

$F(x_q) = q$

By substitution,

$(1 - e^{-k\beta x_q{}^{\lambda} e^{rx_q}})^s = q$ (25)

$(e^{-k\beta x_q{}^{\lambda} e^{rx_q}}) = 1 - q^{\frac{1}{n}}$

$-k\beta x_q^{\lambda} e^{rx_q} = In\left(1 - 0.5^{\frac{1}{n}}\right)$

$x_q^{\lambda} e^{rx_q} = -\frac{1}{k\beta} In\left(1 - q^{\frac{1}{n}}\right)$ (26)

The quantile function is the numerical solution to Equation (26).

Re'nyi Entropy

It is defined by $I_R(\delta) = \frac{1}{1-\delta} \log[I(\delta)]$

Where $I(\delta) = \int_R f^{\delta}(x)dx, \delta > 0$ and $\delta \neq 1$. If X has the EGMW, then

$$I(\delta) = (k\beta s)^{\delta} \int_0^{\infty} x^{\delta(\lambda-1)} (\lambda + rx)^{\delta} e^{r\delta x} e^{(-k\beta x^{\lambda} e^{rx})^{\delta}} (1 - e^{-k\beta x^{\lambda} e^{rx}})^{\delta(s-1)} \qquad (28)$$

Similarly,

$$(1 - e^{-k\beta x^{\lambda} e^{rx}})^{\delta(s-1)} = \sum_{j=0}^{\infty} \binom{\delta(s-1)}{j} (-1)^j e^{-k\beta x^{\lambda} e^{rx} j}$$

Substituting the expression into Equation (28), gives;

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$$I(\delta) = (k\beta s)^{\delta} \sum_{j=0}^{\infty} \binom{\delta(s-1)}{j} (-1)^{j} \int_{0}^{\infty} x^{\delta(\lambda-1)} (\lambda + rx)^{\delta} e^{r\delta x} e^{-k\beta x^{\lambda} e^{rx}(\delta+j)} \quad (29)$$

The exponential function in the integrand becomes,

$$e^{-k\beta x^{\lambda} e^{rx}(\delta+j)} = \sum_{m=0}^{\infty} \frac{(-k\beta(1+j))^{m} x^{\lambda m} e^{rmx}}{m!} \quad (30)$$

By substituting Equation (30) into Equation (29), Equation (29) becomes;

$$= (k\beta s)^{\delta} \sum_{j=0}^{\infty} \binom{\delta(s-1)}{j} (-1)^{j} \sum_{m=0}^{\infty} \frac{(-k\beta(1+j))^{m}}{m!} \int_{0}^{\infty} x^{\delta(\lambda-1)} (\lambda + rx)^{\delta} e^{x(r\delta+rm)} x^{\lambda m}$$

(31)

Solving the integrand in Equation (31);

$$\int_{0}^{\infty} x^{\delta(\lambda-1)} (\lambda + rx)^{\delta} e^{x(r\delta+rm)} x^{\lambda m} = \sum_{k=0}^{\infty} \frac{\binom{s}{k} \lambda^{\delta-k} r^{k} \Gamma(\lambda\delta - \delta + \lambda m + k + 1)}{(-(r^{\delta} + rm))^{\lambda\delta-\delta+\lambda m+k+1}}$$

(32)

Substituting Equation (32) into Equation (31) gives;

$$I(\delta) = (k\beta s)^{\delta} \sum_{j=0}^{\infty} \binom{\delta(s-1)}{j} (-1)^{j} \sum_{m=0}^{\infty} \frac{(-k\beta(1+j))^{m}}{m!} \sum_{k=0}^{\infty} \binom{\delta}{k} \lambda^{\delta-k_{r}k} \frac{\Gamma(\lambda\delta - \delta + \lambda m + k + 1)}{(-(r^{\delta} + rm))^{\lambda\delta-\delta+\lambda m+k+1}}$$

(33)

Hence, the entropy becomes

$$I_R(\delta) = \frac{1}{1-\delta} \log\left[(k\beta s)^{\delta} \sum_{j=0}^{\infty} \binom{\delta(s-1)}{j} (-1)^{j} \sum_{m=0}^{\infty} \frac{(-k\beta(1+j))^{m}}{m!} x \sum_{k=0}^{\infty} \binom{\delta}{k} \lambda^{\delta-k_{r}k} \frac{\Gamma(\lambda\delta - \delta + \lambda m + k + 1)}{(-(r^{\delta} + rm))^{\lambda\delta-\delta+\lambda m+k+1}} \right]$$

(34)

Incomplete moments

This is defined as

$$V_P(z) = \int_0^z x^p f(x)\, dx. \quad (35)$$

Using the relationship in Equation (14),

$$v_p(z) = k\beta s \sum_{j=0}^{\infty} \binom{\delta(s-1)}{j} (-1)^{j} \int_0^z x^{p+\lambda-1} (\lambda + rx) e^{rx} e^{-k\beta x^{\lambda} e^{rx}(1+j)} dx \quad (36)$$

Applying the relationship in Equation (18),

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$$v_p(z) = k\beta s \sum_{j=0}^{\infty} \binom{\delta(s-1)}{j} (-1)^j \sum_{m=0}^{\infty} \frac{(-k\beta(1+j))^m}{m!} \int_0^z x^{p+\lambda-1} (\lambda + rx) e^{rx} e^{-k\beta x^{\lambda} e^{rx}(1+j)} dx$$

(37)

The integrand in Equation (37) gives

$$\int_0^{z^\beta} x^{p+\lambda+\lambda m-1} (\lambda + rx) e^{x(r+rm)} dx = \frac{\lambda}{[-(r+rm)]^{p+\lambda+\lambda m}} \gamma(p+\lambda+\lambda m, z^\beta) + \frac{\lambda}{[-(r+rm)]^{p+\lambda+\lambda m}} \gamma(p+\lambda+\lambda m, z^\beta)$$

(38)

The integrand in Equation (38) is an incomplete gamma. Therefore,

$$v_p(z) = k\beta s \sum_{j=0}^{\infty} \binom{\delta(s-1)}{j} (-1)^j \sum_{m=0}^{\infty} \frac{(-k\beta(1+j))^m}{m!} \left[\frac{\lambda}{[-(r+rm)]^{p+\lambda+\lambda m}} \gamma(p+\lambda+\lambda m, z^\beta) + \frac{r}{[-(r+rm)]^{p+\lambda+\lambda m+1}} \gamma(p+\lambda+\lambda m, z^\beta) \right]$$

(39)

The first incomplete moment is obtained by setting $p = 1$ in Equation (39).

$$V_1(z) = k\beta s \sum_{j=0}^{\infty} \binom{\delta(s-1)}{j} (-1)^j \sum_{m=0}^{\infty} \frac{(-k\beta(1+j))^m}{m!} \left[\frac{\lambda}{[-(r+rm)]^{1+\lambda+\lambda m}} \gamma(1+\lambda+\lambda m, z^\beta) + \frac{r}{[-(r+rm)]^{1+\lambda+\lambda m+1}} \gamma(1+\lambda+\lambda m, z^\beta) \right]$$

(40)

Conditional Moments

The p^{th} conditional moment of EGMW is given in the following theorem.

Theorem 5: Suppose that the random variable X follows the EGMW, then its pth conditional moment is given by

$$E[X^p/X > t] = \frac{k\beta s}{F(t)} \sum_{j=0}^{\infty} \binom{s-1}{j} (-1)^j \frac{(-k\beta(1+j))^m}{m!} x$$

$$\left[A \lambda \left(\Gamma(p+\lambda+\lambda m) - \gamma(p+\lambda+\lambda m, t^\beta\right) + Br(\Gamma(p+\lambda+\lambda m) - \gamma(p+\lambda+\lambda m, t^\beta)\right]$$

Where A = $\left(\frac{1}{-(r+rm)}\right)^{p+\lambda+\lambda m}$ and B = $\left(\frac{1}{-(r+rm)}\right)^{p+\lambda+\lambda m+1}$

Proof

The p^{th} conditional moment is defined by

$$E[X^p/X>t]=\frac{1}{\bar{F}(t)}\int_t^\infty x^p\,f(x)dx.$$

Where $\bar{F}(t)=1-F(t)$. If $f(x)$ is the pdf of EGMW then

$$E[X^p/X>t]=\frac{1}{\bar{F}(t)}\int_t^\infty x^{p+\lambda-1}\,(\lambda+rx)e^{rx}e^{-k\beta x^{\lambda}e^{rx}}\left[1-e^{-k\beta x^{\lambda}e^{rx}}\right]^{s-1}dx \quad (49)$$

$$E[X^p/X>t]=\frac{k\beta s}{F(t)}\sum_{j=0}^{\infty}\binom{s-1}{j}(-1)^j\int_t^\infty x^{p+\lambda-1}\,(\lambda+rx)e^{rx}e^{-k\beta x^{\lambda}e^{rx}(1+j)}dx \quad (50)$$

$$\mathrm{E}[X^p/X>t]=\frac{k\beta s}{\bar{F}(t)}\sum_{j=0}^{\infty}\binom{s-1}{j}(-1)^j\sum_{m=0}^{\infty}\frac{\left(-k\beta(1+j)\right)^m}{m!}\int_t^\infty x^{p+\lambda+\lambda m-1}(\lambda+rx)e^{x(r+rm)}dx \quad (51)$$

Consider the integrand in Equation (51)

$$\int_t^\infty x^{p+\lambda+\lambda m-1}(\lambda+rx)e^{x(r+rm)}dx=\lambda\int_t^\infty x^{p+\lambda+\lambda m-1}e^{x(r+rm)}dx+r\int_t^\infty x^{p+\lambda+\lambda m-1}e^{x(r+rm)}dx \quad (52)$$

but

$$\lambda\int_t^\infty x^{p+\lambda+\lambda m-1}e^{x(r+rm)}dx=\lambda\left(\frac{1}{-(r+rm)}\right)^{p+\lambda+\lambda m}\int_t^\infty y^{p+\lambda+\lambda m-1}e^{-y}dy \quad (53)$$

and

$$r\int_t^\infty x^{p+\lambda+\lambda m-1}e^{x(r+rm)}dx=\mathrm{r}\left(\frac{1}{-(r+rm)}\right)^{p+\lambda+\lambda m+1}\int_t^\infty y^{p+\lambda+\lambda m}\,e^{-y}dy \quad (54)$$

The integrand in Equation (53) can be expressed as

$$\int_t^\infty y^{p+\lambda+\lambda m-1}e^{-y}dy=\int_0^\infty y^{p+\lambda+\lambda m-1}e^{-y}dy-\int_0^{t^\beta}x^{p+\lambda+\lambda m-1}e^{-y}dy \quad (55)$$

The first integrand is a complete gamma function and the second integrand is an incomplete gamma function. Therefore,

$$\int_0^\infty y^{p+\lambda+\lambda m-1}e^{-y}dy-\int_0^{t^\beta}x^{p+\lambda+\lambda m-1}e^{-y}dy=\Gamma(p+\lambda+\lambda m)-\gamma\left(p+\lambda+\lambda m,t^\beta\right) \quad (56)$$

Similarly, the integrand in Equation (54) also gives

$$\int_0^\infty y^{p+\lambda+\lambda m}e^{-y}dy-\int_0^{t^\beta}y^{p+\lambda+\lambda m}e^{-y}dy=\Gamma(p+\lambda+\lambda m)-\gamma\left(p+\lambda+\lambda m+1,t^\beta\right) \quad (57)$$

Hence (52) becomes

$$\int_t^\infty x^{p+\lambda+\lambda m-1}(\lambda+rx)e^{x(r+rm)}dx=[A\lambda\,(\Gamma(p+\lambda+\lambda m)-\gamma\left(p+\lambda+\lambda m,t^\beta\right)+$$

$$\beta r\left(\Gamma(p+\lambda+\lambda m+1)-\gamma\left(p+\lambda+\lambda m+1,t^\beta\right)\right] \quad (58)$$

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Substituting Equation (58) into Equation (51) gives the conditional moment of the distribution.

$$E[X^p/X>t]=\frac{k\beta s}{\bar{F}(t)}\sum_{j=0}^{\infty}\binom{s-1}{j}(-1)^j\frac{(-k\beta(1+j))^m}{m!}\ x$$

$$\left[A\curlywedge(\Gamma(p+\curlywedge+\curlywedge m)-\gamma(p+\curlywedge+\curlywedge m,t^{\beta})+Br(\Gamma(p+\curlywedge+\curlywedge m)-\gamma(p+\curlywedge+\curlywedge m,t^{\beta})\right]$$

Where A= $\left(\frac{1}{-(r+rm)}\right)^{p+\curlywedge+\curlywedge m}$ and B = $\left(\frac{1}{-(r+rm)}\right)^{p+\curlywedge+\curlywedge m+1}$

The Order Statistics

The pdf of order statistics of EGMW distribution is defined as

$$fx_{pin}=\frac{n!}{(p-1)!(n-p)!}f_x(x)(F_x(x))^{p-1}(1-F_x(x))^{n-P}. \quad (59)$$

Substituting the probability density function and cumulative density function of the EGMW distribution into (59) gives;

$$fx_{pin}=\frac{n!}{(p-1)!(n-p)!}\ k\beta s x^{\curlywedge-1}(\curlywedge+rx)e^{rx}e^{-k\beta x^{\curlywedge}e^{rx}}\left[1-e^{-k\beta x^{\curlywedge}e^{rx}}\right]^{s-1}$$

$$X\ ((1-e^{-k\beta x^{\curlywedge}}e^{rx})^s)^{P-1}(1-(1-e^{-k\beta x^{\curlywedge}e^{rx}})^s)^{n-p} \quad (60)$$

where P= 1, 2, . . . n.

The probability density function of the maximum order statistics when p = n is given by

$$fx_{nin^{(X)}}=nk\beta s x^{\curlywedge-1}(\curlywedge+rx)e^{rx}e^{-k\beta x^{\curlywedge}e^{rx}}(1-e^{-k\beta x^{\curlywedge}}e^{rx})^{s-1}((1-e^{-k\beta x^{\curlywedge}}e^{rx})^s)^{n-1}$$

(61)

The probability density function of the minimum order statistic when p = 1 is given by $fx_{nin^{(X)}}=nk\beta s x^{\curlywedge-1}(\curlywedge+rx)e^{rx}e^{-k\beta x^{\curlywedge}e^{rx}}(1-e^{-k\beta x^{\curlywedge}}e^{rx})^{s-1}(1-(1-e^{-k\beta x^{\curlywedge}}e^{rx})^s)^{n-1}$ (62)

Parameter Estimation

Suppose $X_{1,\ldots,}X_n$ are independent random variables with sample size n from Exponentially Generated Modified Weibull distribution, its likelihood function is given by

$$\ell=(k\beta s)^n e^{r\sum_{i=1}^{n}x_i}e^{-k\beta\sum_{i=1}^{n}x_i^{\curlywedge}e^{rx_i}}\prod_{i=1}^{n}x_i^{\curlywedge-1}(\curlywedge+rx_i)(1-e^{-k\beta x^{\curlywedge}}e^{rx_i})^{s-1} \quad (63)$$

its log-likelihood function is given by

$$\ln\ell=n\ln(k\beta s)+r\sum_{i=1}^{n}x_i-k\beta\sum_{i=1}^{n}x_i^{\curlywedge}e^{rx_i}+(\curlywedge-1)\sum_{I=1}^{n}\ln x_I+\sum_{I=1}^{n}\ln(r+x_i)+(s-1)\sum_{I=1}^{n}\ln(1-e^{-k\beta x_i^{\curlywedge}\quad tax_i}) \quad (64)$$

The estimates of parameters β, k, r, ⅄ and s are obtained by taking the derivative of the log-likelihood function concerning each parameter and equating it to zero. The following system of equations were obtained;

$$\frac{d\ell}{d\beta}=\frac{n}{\beta}-k\sum_{i=1}^{n}x_i^{\curlywedge}e^{rx_i}+k(s-1)\frac{\sum_{i=1}^{n}x_i^{\curlywedge}e^{rx_i}e^{-k\beta x_i^{\curlywedge}e^{rx_i}}}{1-e^{-k\beta x_i^{\curlywedge}e^{rx_i}}}=0 \quad (65)$$

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$$\frac{d\ell}{dk} = \frac{n}{k} - \beta\sum_{i=1}^{n} x_i^{\lambda} e^{rx_i} + \beta(s-1)\frac{\sum_{i=1}^{n} x_i^{\lambda} e^{rx_i} e^{-k\beta x_i^{\lambda} e^{rx_i}}}{1-e^{-k\beta x_i^{\lambda} e^{rx_i}}} = 0 \quad (66)$$

$$\frac{d\ell}{ds} = \frac{n}{s} + \sum_{i=1}^{n} In\left(1 - e^{-k\beta x_i^{\lambda} e^{rx_i}}\right) = 0 \quad (67)$$

$$\frac{d\ell}{dr} = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\frac{x_i}{\lambda + rx_i} - k\beta\sum_{i=1}^{n} x_i^{\lambda-1} e^{rx_i} + k\beta(s-1)\frac{\sum_{i=1}^{n} x_i^{\lambda} e^{rx_i} e^{-k\beta x_i^{\lambda} e^{rx_i}}}{1-e^{-k\beta x_i^{\lambda} e^{rx_i}}} = 0 \quad (68)$$

$$\frac{d\ell}{d\lambda} = \sum_{i=1}^{n} In x_i + r\sum_{i=1}^{n}(\lambda + rx_i)^{-1} - k\beta\sum_{i=1}^{n} x_i^{\lambda-1} e^{rx_i} + k\beta(s-1)\frac{\sum_{i=1}^{n} x_i^{\lambda} e^{rx_i} e^{-k\beta x_i^{\lambda} e^{rx_i}}}{1-e^{-k\beta x_i^{\lambda} e^{rx_i}}} = 0 \quad (69)$$

The above equations are non-linear in parameters. A numerical optimization method can be used to obtain the inherent parameters.

Simulation Study of the Exponentiated Generated Modified Weibull Distribution

This is necessary to determine the homogeneity property of the generalized distribution. For the distribution to be stable the estimates must be closer to the true values of the parameters, otherwise, the distribution is not stable. Using the Monte Carlo simulation approach, three replicates were formed at different sample sizes; 50, 100, 200, and 500. Among the parameters in the model, in the first group, β was fixed at 0.5 and λ at 0.5. In the second category, β was fixed at 1.0 and λ at 0.5. In the third category, β was fixed at 0.5 and λ at 1.0. For all the replicates, s was fixed at 0.1, k at 2.0, and r at 0.1. From the output in Table 1, it can be observed that as sample size increases, the biasness of the estimates reduces leading to a significant reduction in the Mean Square Error of the estimates. Also, it can be observed that an increase in sample sizes lead to the closeness of the estimates to the true value of the parameters. Therefore, it can be concluded that the distribution is stable.

Table 1: Simulation Output of EGMW Distribution for Varying Parameter Values

s = 0.1, k = 2.0, r = 0.1		**β = 0.5, λ = 0. 5**			**β = 1.0, λ = 0. 5**			**β = 0.5, λ = 1. 0**		
N	**Par**	**Est**	**BIAS**	**MSE**	**Est**	**BIAS**	**MSE**	**Est**	**BIAS**	**MSE**
50	β	0.71523	0.21523	0.04650	1.13644	0.13644	0.01862	0.02841	0.47159	0.22241
	λ	0.48466	0.01534	0.00025	0.48761	0.01239	0.00015	0.95415	0.04585	0.00302
	S	0.09548	0.00452	0.00002	0.09155	0.00845	0.00007	0.09129	0.00871	0.00008
	k	2.00303	0.00303	0.00002	2.03629	0.03629	0.00132	1.76726	0.23274	0.49247
	r	0.04053	0.05947	0.00358	0.02922	0.07078	0.00502	0.01313	0.08687	0.00766
N	**Par**	**Est**	**BIAS**	**MSE**	**Est**	**BIAS**	**MSE**	**Est**	**BIAS**	**MSE**
100	β	0.73753	0.23753	0.05672	1.07459	0.07459	0.00557	0.01829	0.48171	0.23222
	λ	0.47816	0.02184	0.00048	0.48819	0.01181	0.00014	0.57582	0.42418	0.26486
	s	0.09358	0.00642	0.00004	0.09184	0.00816	0.00007	0.06532	0.03468	0.00160
	k	2.00512	0.00512	0.00003	2.02565	0.02565	0.00066	3.47920	1.47920	3.05754

	r	0.02287	0.07713	0.00604	0.03815	0.06185	0.00387	0.00871	0.09129	0.00839
N	**Par**	**Est**	**BIAS**	**MSE**	**Est**	**BIAS**	**MSE**	**Est**	**BIAS**	**MSE**
200	β	0.70361	0.20361	0.04338	1.06057	0.06057	0.00369	0.04702	0.45298	0.20592
	λ	0.47822	0.02178	0.00050	0.48277	0.01723	0.00030	0.86761	0.13239	0.03383
	s	0.09336	0.00664	0.00005	0.08942	0.01058	0.00011	0.08508	0.01492	0.00029
	k	2.00701	0.00701	0.00005	2.02393	0.02393	0.00058	2.33738	0.33738	0.61982
	r	0.02283	0.07717	0.00614	0.02134	0.07866	0.00630	0.01607	0.08393	0.00720
N	**Par**	**Est**	**BIAS**	**MSE**	**Est**	**BIAS**	**MSE**	**Est**	**BIAS**	**MSE**
500	β	0.65577	0.15577	0.02563	1.03566	0.03566	0.00136	0.06011	0.43989	0.19538
	λ	0.48026	0.01974	0.00041	0.48268	0.01732	0.00030	0.88523	0.11477	0.01847
	s	0.09357	0.00643	0.00004	0.09107	0.00893	0.00008	0.08573	0.01427	0.00025
	k	2.00874	0.00874	0.00008	2.01521	0.01521	0.00024	2.54293	0.54293	0.56739
	r	0.02651	0.07349	0.00558	0.02926	0.07074	0.00508	0.01402	0.08598	0.00743

Model Comparison

As part of the performance test, there is a need to compare the distribution with existing ones in the same category using secondary data. Two cases were used using data from research of other researchers. See the output below;

The data is on the breaking stress of carbon fibres of 50 mm length (GPa). The data has been previously used by Nichols and Padgett [15], Cordeiro and Lemonte [8], Al-Aqtash *et al*. [3], and Oguntunde *et al*. [16]. The data is as follows:

Table 2: Breaking Strength of Carbon Fibres of 50mm Length (GPa)

0.39	0.85	1.08	1.25	1.47	1.57	1.61	1.61	1.69	1.80
1.84	1.87	1.89	2.03	2.03	2.05	2.12	2.35	2.41	2.43
2.48	2.50	2.53	2.55	2.55	2.56	2.59	2.67	2.73	2.74
2.79	2.81	2.82	2.85	2.87	2.88	2.93	2.95	2.96	2.97
3.09	3.11	3.11	3.15	3.15	3.19	3.22	3.22	3.27	3.28
3.31	3.31	3.33	3.39	3.39	3.56	3.60	3.65	3.68	3.70
3.75	4.20	4.38	4.42	4.70	4.90				

Table 3: Output of Analysis of the data on Existing and Newly Generalized Distribution

Distributions	Parameter Estimates	Log-Likelihood	AIC	Rank
Weibull Exponential	$\alpha = 5.25929$ $\beta = 2.80643$ $\lambda = 0.14236$	-85.7833	171.5667	2
Exponentiated Generalized Modified Weibull Distribution (EGMW)	β = 0.02360 λ = 0.14926 S = 5.47633 K = 18.41287 r = 0.51670	-85.3731	170.7462	1

From Table 3, the proposed distribution, EGMW distribution, has a higher log-likelihood value and lower AIC value compared with the Weibull Exponential distribution. Therefore, it can be concluded that the EGMW distribution modeled the data better than the distribution compared with.

Case II

Using a data set previously used by Ahmed et al. [1] on a length of 10mm from Kandu and Raqab [10]. The data set consists of 63 observations

Table 4: Lenght of 10mm Extracted from work of Kanda and Raqab [10]

1.901	2.132	2.203	2.228	2.257	2.35	2.361	2.396	2.397	2.445
2.454	2.474	2.518	2.522	2.525	2.532	2.575	2.614	2.616	2.618
2.624	2.659	2.675	2.738	2.74	2.856	2.917	2.928	2.937	2.937
2.977	2.996	3.03	3.125	3.139	3.145	3.22	3.223	3.235	3.243
3.264	3.272	3.294	3.332	3.346	3.377	3.408	3.435	3.493	3.501
3.537	3.554	3.562	3.628	3.852	3.871	3.886	3.971	4.024	4.027
4.225	5.395	5.02							

It was used to show the suitability and superiority of Transmitted Weibull-Pareto (TWPa) distribution over Weibull Pareto (WPa), Transmuted Weibull Lonax (TWL), Transmutted Complimentary Weibull (TCW) and McDonald Lomax (McL) distributions. Using the data for modeling, the output is as shown below;

Table 5: Output of the Analysis of 10mm data modeling

Models	Estimates	Log-Likelihood	AIC	BIC
EGMW	β = 1.3067 λ = 1.0509 S = 149.0130 K = 1.3067	-58.4003	126.80007	137.5163

	r = 0.0010			
TWPa	a=0.1885 b=0.0909 c=14.4535 d=0.7280		127.282	
Wpa	a=0.1834 b=0.0755 c=13.9522		127.790	
TWL	a=0.3922 b=0.6603 c=0.5287 d=8.4451 e=0.7364		129.688	
McL	a=45.9249 b=48.3024 c=18.1192 d=195.4633 e=353.1435		140.597	
TCW	a=0.2022 b=3.3482 c=0.3076 d=-0.0001		134.895	

Table 5 shows the superiority of the Exponentiated Generalized Modified Weibull distribution over five distributions. It can be deduced that in modeling the data extracted from the work of Ahmed et al. [1] on a length of 10mm, the most appropriate model is the Exponentiated Generalized Modified Weibull distribution as it has the lowest AIC value among the distributions compared with.

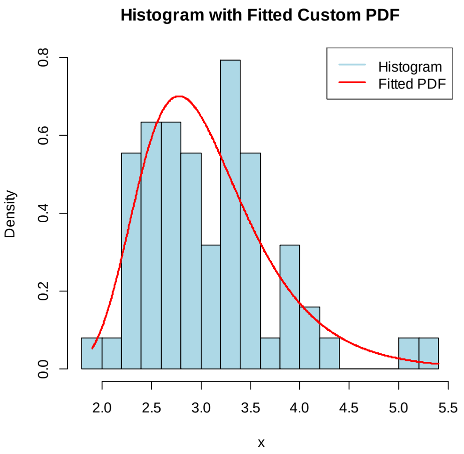

Figure 5: Histogram with fitted line on the data.

SUMMARY OF FINDINGS AND CONCLUSION

In this paper, the Exponentiated approach was used as a method of generalization of distribution using the cumulative density function of a well-known distribution which served as the baseline function. For the generalization of a distribution for better performance, a modified Weibull distribution with three parameters was used as a baseline function which resulted in a five-parameter modified Weibull distribution called Exponentiated Generated Modified Weibull (EGMW) distribution. The generator proposed by Cordairo et al. [9] led to a cumulative density function from which a probability density function was formed. The newly generated distribution, EGMW, was tested for completeness using one of the properties of a proper probability density function. The test of completeness was done using the property called area under the curve and it yielded 1 as a result of the integral of the probability density function.

To buttress the point, statistical properties of the EGMW were studied which include moment, moment generating function, characteristic function, median, quantile function, mean, variance, mean deviation, incomplete moment, Lorenz and Bonferroni curves, conditional moments, and order statistics. Parameters in the formulated model were estimated using the method of Maximum Likelihood.

Graph of a probability density function, cumulative density function, survival function, and hazard function of the distribution were plotted using different parameter values. Also, the Monte Carlo simulation approach was used for the study of the stability (homogeneity) of the distribution. In the simulation, three replicates were used at varying parameter values and different sample sizes of 50, 100, 200, and 500. Biasness and Mean Square Error (MSE) were used for the appropriateness of the estimates. It was observed that the estimates approach true values of the parameters as sample size increases which leads to a significant reduction in the biasness and MSE. Based on these facts, it was concluded that the resulting distribution is stable and can be used for modeling.

For more fact-finding, the newly generalized distribution was compared with existing distributions in its category using secondary data. Two cases were considered involving data on breaking stress of carbon fibres of 50 mm length (GPa), previously used by Nicholis and Padgett [15], Cordeiro and Lemonte [8], Al-Aqtash et al [3] and Oguntunde *et al*. [16] and data on length of 10mm rod previously used by Kandu and Raqab [10] and Ahmed et al. [1]. In the ranking of the distributions, for the first case, AIC and Loglikelihood were used. In the second case, AIC was used for the rating. It was observed that EGMW performed better than the distributions compared with, in the two cases involving different sets of data. Therefore, the newly generated distribution is recommended for usage in studies involving the probability density function of its category.

REFERENCES

1. Ahmed Z. A., Zohdi M. N., Haitham M. Y., Yehia M. G. and Nadeem S. B (2015); The Transmuted Weibull Lomax Distribution: Properties and Application. Pakistan Journal of Statistics and Operation Research. Vol.XI, No.1, pp151-168
2. Aldahlan, M.A., Afify, A.Z., 2020. The odd exponentiated half-logistic exponential distribution: estimation methods and application to engineering data. Mathematics 8 (10), 1684.
3. Al-Aqtash, R., C. Lee and F. Famoye (2014). Gumbel-Weibull distribution: Properties and applications. J. Mod. Applied Stat. Methods, 13: 201-225
4. Awopeju K. A. and Alfred A. A. (2023). Transmuted Modified Weibull Distribution for Modeling Skewed Lifetime Dataset; Properties and Application. Journal of Probability and Statistical Science. 21(2), 36-61.
5. Bebbington, M., Lai, C. D. and Zitikis, R. (2007).A Flexible Weibull Extension. Reliability Engineering and System Safety, 92, 719-726.
6. Carrasco, J. M. F., Ortega, E. M. M. and Cordeiro, G. M. (2008).A generalized modified Weibull distribution for lifetime modeling. Computational Statistics and Data Analysis, 53, 450-462.
7. Chen, Z. (2000): A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function. Statistics and Probability Letters, **49(2)**, 155-161.
8. Cordeiro, G.M. and A.J. Lemonte (2011). The β-Birnbaum-Saunders distribution: An improved distribution for fatigue life modeling. Comput. Stat. Data Anal., 55: 1445-1461.
9. Cordeiro, G.M., Ortega, G.M., and Cunha, D.C. (2013). The exponentiated generalized class of distributions, Journal of Data Science,**11**, 1 - 27.
10. Kandu, D. and Raqab, M. Z. (2009). Estimation of R = P (Y < X) for three-parameter Weibull distribution. Statistics and Probability Letters. Vol. 79, 1839-1846. [8] Lai C, Xie M, and Murthy D (2003) A modified Weibull distribution. IEEE Trans Reliab 52(1):33–37.
11. Lai C, Xie M, and Murthy D (2003) A modified Weibull distribution. IEEE Trans Reliab 52(1):33–37.
12. Lawless, J. F. (1982).Statistical Models and Methods for Lifetime Data. New York: John Wiley and Sons

13. Marshall A. N., and Olkin, I..(1997). A New Method for Adding a Parameter to a Family of Distributions with Applications to the Exponential and Weibull Families, Biometrics 84, 641-652.
14. Mudholkar, G. S. and Srivastava, D. K. (1993).Exponentiated Weibull family for analyzing bathtub failure-rate data.IEEE Transactions on Reliability, **42**(2): 299-302.
15. Nichols, M.D. and W.J. Padgett (2006): A bootstrap control chart for Weibull percentiles. Quality Reliable. Eng. Int., 22: 141-151.
16. Oguntunde P. E., Balogun O. S., Okagbue H. I., and Bishop S. A. (2015): The Weibull-Exponential Distribution: Its Properties and Applications. Journal of Applied Sciences, 15; 1305-1311.
17. Xie M, TangY.and Goh T. N. (2002). A Modified Weibull Extension with Bathtub Failure Rate Function. ReliabEngSystSaf 76:279-285.
18. Zhang, T. and Xie, M. (2011) On the Upper Truncated Weibull Distribution and Its Reliability Implications. Reliability Engineering and System Safety, 96, 194-200.