

Gamma-Exponential-Exponential Distribution (Geed): Theories and Properties

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ABSTRACT

The paper proposes a new family of continuous distributions called the Gamma-Exponential-Exponential distribution (GEED). It is based on the $T X$ construction of Alzaatreh *et al.* (2013) by considering Gamma-Exponential distribution for T and the exponential as the distribution of X . Some of useful mathematical and statistical properties such as the moment method, moment generating function, characteristics function skewness and kurtosis, are derived, and a number of important statistical characteristics of this family are investigated the method of maximum likelihood estimation is used to estimate the model parameters. The usefulness of this family of distributions is demonstrated via simulated experiments aiming to assess their performances.

Keywords: Gamma-Exponential-Exponential, Gamma-Exponential, Moment, Maximum Likelihood Estimator, Simulation

INTRODUCTION

Statistical distributions are very useful in describing real-world phenomenon. Although many distributions have been developed but there is always a room for new distributions which are more flexible in term of fitting a specific real-world scenario. This attempt has motivated researchers to seek and develop new flexible distributions. As a result, many new distributions have been developed and studied, from the past years, there is a growing trend of generating new families of distributions from existing distribution by adding one or more additional parameter(s) to the baseline distribution to study the behavior of the shapes of density and hazard rate, and for checking the goodness-of-fit, we intend to introduce Gamma-Exponential-Exponential distribution (GEED), a novel member of the Gamma-X family of generalized distributions, development on generating new distribution to model naturally occurring phenomena have led to a number of new distributions being defined and studied. Some of the earlier works include those of Alzaatreh *et al.* (2013), Malik *et al.* (2019), Hussein *et al.* (2023), Jaroengeratikun *et al.* (2022), Akarawak *et al.* (2015), Akarawak *et al.* (2017), Ayeni *et al.* (2023), Ogunwale *et al.* (2019) and Ogunwale *et al.* (2024) are a few of the previous works. Additionally, a number of researches (Mudholkar and Serivastava, 1993; Gupta and Kundu, 2001; Pal *et al.*, 2006) have demonstrated the superior performance and flexibility of mixed random variable distributions. Due to this, we plan to create a new continuous probability distribution called the Gamma-Exponential-Exponential distribution by combining three different continuous probability distributions. This distribution will be developing using the Transform-transformer technique and will offer methods for generalizing a more manageable and robust probability distribution to cater for the deficiencies of single pdf distribution.

METHODS

Formulation of Gamma- Exponential –X distribution

Let x be continuous independent random variables such that; $x \sim \text{GED}(x, \sigma, \beta, \lambda)$ and, let $f(x)$ and $F(x)$ be the probability density function and Cumulative distribution function of Exponential- Gamma distribution given as;

$$f(x) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x/\beta}}{\Gamma(\alpha) \beta^\alpha}, x >, \alpha, \beta, \lambda > 0 \tag{1}$$

$$F(x) = \frac{\gamma(x, \alpha)}{\Gamma(\alpha)} \tag{2}$$

Using the Transform-Transformer Techniques to merge Gamma-Exponential-X

$$G(x) = \frac{F(x)}{1-F(x)} r(x) \text{ where } t = -\log(1-f(x)) \tag{3}$$

$$r(x) = \frac{\lambda^\alpha t^{\alpha-1} e^{-\frac{\lambda t}{\beta}}}{\sqrt{\alpha} \beta^\alpha} \tag{4}$$

$$r(x) = \frac{\lambda^\alpha [-\log(1-f(x))]^{\alpha-1}}{\sqrt{\alpha} \beta^\alpha} e^{-\lambda(-\log(1-f(x))) \cdot \frac{1}{\beta}} \tag{5}$$

$$r(x) = \frac{\lambda^\alpha [-\log(1-f(x))]^{\alpha-1}}{\sqrt{\alpha} \beta^\alpha} e^{-\frac{\lambda}{\beta}(-\log(1-f(x)))} \tag{6}$$

$$r(x) = \frac{\lambda^\alpha [-\log(1-f(x))]^{\alpha-1} (1-f(x))^{\lambda/\alpha}}{\sqrt{\alpha} \beta^\alpha} \tag{7}$$

$$G(x) = \frac{f(x)}{1-f(x)} \cdot \lambda^\alpha [-\log(1-f(x))]^{\alpha-1} (1-f(x))^{\lambda/\beta} \cdot \frac{1}{\sqrt{\alpha} \beta^\alpha} \tag{8}$$

$$G(x) = \frac{f(x)}{\sqrt{\alpha} \beta^\alpha} \cdot \lambda^\alpha [-\log(1-f(x))]^{\alpha-1} (1-f(x))^{\lambda/\beta-1} \tag{9}$$

$$G(x) = \frac{f(x)}{\sqrt{\alpha} \beta^\alpha} \lambda^\alpha [-\log(1-f(x))]^{\alpha-1} (1-f(x))^{\lambda/\beta-1} \tag{10}$$

$$G(x) = \frac{\lambda e^{-\lambda x}}{\sqrt{\alpha} \beta^\alpha} \lambda^\alpha [-\log(1-f(x))]^{\alpha-1} (1-f(x))^{\lambda/\beta-1} \tag{11}$$

$$F(x) = \int_0^\infty f(x) dx = \int_0^\infty \lambda e^{-\lambda x} = \lambda \int_0^\infty e^{-\lambda x} dx = 1 - e^{-\lambda x}$$

$$= \frac{\lambda e^{-\lambda x}}{\sqrt{\alpha} \beta^\alpha} \lambda^\alpha [-\log(1-f(x))]^{\alpha-1} (1-f(x))^{\lambda/\beta-1} \tag{12}$$

$$= \frac{\lambda e^{-\lambda x}}{\sqrt{\alpha} \beta^\alpha} \lambda^\alpha [-\log(1-1+e^{-\lambda x})]^{\alpha-1} (1-1+e^{-\lambda x})^{\lambda/\beta-1} \tag{13}$$

$$= \frac{\lambda e^{-\lambda x}}{\sqrt{\alpha} \beta^\alpha} \lambda^\alpha [-\log e^{-\lambda x}]^{\alpha-1} (e^{-\lambda x})^{\lambda/\beta-1} \tag{14}$$

$$= \frac{\lambda e^{-\lambda x}}{\sqrt{\alpha \beta^\alpha}} \lambda^\alpha (\lambda x)^{\alpha-1} (e^{-\lambda x})^{\lambda/\beta-1} \tag{15}$$

$$= \frac{\lambda e^{-\lambda x}}{\sqrt{\alpha \beta^\alpha}} \lambda^\alpha (\lambda x)^{\alpha-1} (e^{\lambda x} - \frac{\lambda^2 x}{\beta}) \tag{16}$$

$$= \frac{\lambda e^{-\lambda x}}{\sqrt{\alpha \beta^\alpha}} \lambda^\alpha \lambda^{\alpha-1} x^{\alpha-1} e^{\lambda x} e^{-\frac{\lambda^2 x}{\beta}} \tag{17}$$

The probability density function of Gamma-Exponential- Exponential distribution (GEED) is given as

$$= \frac{\lambda^{2\alpha} x^{\alpha-1} e^{-\frac{\lambda^2 x}{\beta}}}{\sqrt{\alpha \beta^\alpha}} \tag{18}$$

Statistical Properties

This section presents the foundational statistical properties of the Gamma-Exponential Distribution (GED), focusing specifically on the first four moments, variance, and coefficient of variation, moment-generating function, characteristic function, skewness, and kurtosis.

(i) Moments

Theorem 1: if X is a random then the r^{th} non- central moment is given by:

$$\mu_r' = \int_0^\infty x^r f(x, \alpha, \beta) dx \tag{19}$$

$$\text{let } u = \frac{\lambda^2 x}{\beta}$$

$$x = \frac{u\beta}{\lambda^2}$$

$$dx = \frac{\beta du}{\lambda^2}$$

$$\mu_r' = \int_0^\infty \frac{x^r \lambda^{2\alpha} x^{\alpha-1} e^{-\frac{\lambda^2 x}{\beta}}}{\sqrt{\alpha \beta^\alpha}} dx \tag{20}$$

$$\mu_r' = \int_0^\infty \frac{\lambda^{2\alpha} x^{r+\alpha-1} e^{-\frac{\lambda^2 x}{\beta}}}{\sqrt{\alpha \beta^\alpha}} dx \tag{21}$$

$$\mu_r' = \int_0^\infty \left(\frac{\lambda\beta}{\lambda^2}\right)^{r+\alpha-1} \cdot \lambda^{2\alpha} e^{-\lambda} \cdot \frac{\beta du}{\lambda^2} \cdot \frac{1}{\sqrt{\alpha \beta^\alpha}} \tag{22}$$

$$\mu_r' = \frac{1}{\sqrt{\alpha \beta^\alpha}} \int_0^\infty \beta^{r+\alpha-1} \cdot u^{r+\alpha-1} \cdot \frac{1}{\lambda^{2r+2\alpha-2}} \cdot \lambda^{2\alpha} \cdot \frac{1}{\lambda^2} \cdot e^{-u} \cdot \beta du \tag{23}$$

$$\mu'_r = \frac{1}{\sqrt{\alpha\beta^\alpha}} \int_0^\infty \beta^{r+\alpha} \cdot u^{r+\alpha-1} \cdot \frac{1}{\lambda^{2r+2\alpha-2}} \cdot \lambda^{2\alpha} \cdot \frac{1}{\lambda^2} \cdot e^{-u} du \tag{24}$$

$$\mu'_r = \frac{1}{\sqrt{\alpha\beta^\alpha}} \int_0^\infty \beta^{r+\alpha} \cdot u^{r+\alpha-1} \cdot \frac{1}{\lambda^{2r-2}} \cdot \frac{1}{\lambda^2} \cdot e^{-u} du \tag{25}$$

$$\mu'_r = \frac{1}{\sqrt{\alpha\beta^\alpha}} \int_0^\infty u^{r+\alpha-1} \cdot e^{-u} \cdot \beta^{r+\alpha} \cdot \frac{1}{\lambda^{2r}} du \tag{26}$$

$$\mu'_r = \frac{\beta^r}{\sqrt{\alpha\lambda^{2r}}} \cdot \Gamma(\alpha+r)$$

(27)

From (27), the first (mean), second, third and the fourth moments shall be obtained respectively as follows; Substituting $r = 1$, then the Mean is obtained as;

$$\mu'_1 = \frac{\beta\Gamma(\alpha+1)}{\lambda^2\Gamma^\alpha} = \frac{\beta^\alpha\Gamma^\alpha}{\lambda^2\Gamma^\alpha} = \frac{\alpha\beta}{\lambda^2} \tag{28}$$

$$\mu'_2 = \frac{\beta^2\Gamma(\alpha+2)}{\lambda^4\Gamma^\alpha} = \frac{\alpha(\alpha+1)\beta^2\Gamma^\alpha}{\lambda^4\Gamma^\alpha} = \frac{\alpha(\alpha+1)\beta^2}{\lambda^4} \tag{29}$$

$$\mu'_3 = \frac{\beta^3\Gamma(\alpha+3)}{\lambda^6\Gamma^\alpha} = \frac{\alpha(\alpha+1)(\alpha+2)\beta^3\Gamma^\alpha}{\lambda^6\Gamma^\alpha} = \frac{\alpha(\alpha+1)(\alpha+2)\beta^3}{\lambda^6} \tag{30}$$

$$\mu'_4 = \frac{\beta^4\Gamma(\alpha+4)}{\lambda^8\Gamma^\alpha} = \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)\beta^4\Gamma^\alpha}{\lambda^8\Gamma^\alpha} = \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{\lambda^8} \tag{31}$$

$$V(X) = \mu'_1 - \mu_2$$

$$V(x) = \frac{\alpha(\alpha+1)\beta^2}{\lambda^4} - \left(\frac{\alpha\beta}{\lambda^2}\right)^2 \tag{32}$$

$$V(x) = \frac{\alpha(\alpha+1)\beta^2}{\lambda^4} - \frac{d^2\beta^2}{\lambda^4}$$

$$V(x) = \frac{\alpha(\alpha+1)\beta^2 - \alpha^2\beta^2}{\lambda^4} \tag{33}$$

$$V(x) = \frac{\alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2}{\lambda^4} \tag{34}$$

$$V(x) = \frac{\alpha\beta^2}{\lambda^4}$$

(35)

Moment Generating Function (M.G.F)

Theorem 2: If X is a continuous random variable distributed as a GEED $(x, \lambda, \beta, \alpha)$ then the moment generating function is given as $\frac{\lambda^2}{\beta (t - \frac{\lambda^2}{\beta})^\alpha}$

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x, \alpha, \beta, \lambda) dx$$

$$M_x(t) = \int_0^\infty \frac{e^{tu} \lambda^{2\alpha} x^{\alpha-1} e^{-\frac{\lambda^2 x}{\beta}}}{\sqrt{\alpha\beta}^\alpha} dx \tag{36}$$

$$M_x(t) = \frac{1}{\sqrt{\alpha\beta}^\alpha} \int_0^\infty \lambda^{2\alpha} x^{\alpha-1} e^{tx} \cdot \frac{\lambda^{2x}}{\beta} dx \tag{37}$$

$$M_x(t) = \frac{1}{\sqrt{\alpha\beta}^\alpha} \int_0^\infty \lambda^{2\alpha} x^{\alpha-1} e^{x(\frac{t-\lambda^2}{\beta})} dx \tag{38}$$

Let $u = x(\frac{t-\lambda^2}{\beta})$, $x = \frac{u}{\frac{t-\lambda^2}{\beta}}$, $dx = \frac{du}{\frac{t-\lambda^2}{\beta}}$

$$M_x(t) = \frac{\lambda^{2\alpha}}{\sqrt{\alpha\beta}^\alpha} \int_0^\infty \left(\frac{u}{\frac{t-\lambda^2}{\beta}}\right)^{\alpha-1} \cdot e^u \cdot \frac{du}{\frac{t-\lambda^2}{\beta}} \tag{39}$$

$$M_x(t) = \frac{\lambda^{2\alpha}}{\sqrt{\alpha\beta}^\alpha} \int_0^\infty u^{\alpha-1} \cdot \left(\frac{1}{\frac{t-\lambda^2}{\beta}}\right)^{\alpha-1} \cdot \frac{1}{\frac{t-\lambda^2}{\beta}} e^u du \tag{40}$$

$$M_x(t) = \frac{\lambda^{2\alpha}}{\sqrt{\alpha\beta}^\alpha} \int_0^\infty u^{\alpha-1} \cdot \left(\frac{1}{\frac{t-\lambda^2}{\beta}}\right)^\alpha \cdot e^u du$$

$$M_x(t) = \frac{\lambda^{2\alpha}}{\sqrt{\alpha\beta}^\alpha (t - \frac{\lambda^2}{\beta})^\alpha} \tag{41}$$

$$M_x(t) = \frac{\lambda^{2\alpha}}{\beta^\alpha (t - \frac{\lambda^2}{\beta})^\alpha} \text{ Or } \left[\frac{\lambda^2}{\beta (t - \frac{\lambda^2}{\beta})} \right]^\alpha$$

Characteristic Function (C.F)

Theorem 3: If X is a random variable distributed as a GE ED $(x; \alpha, \beta, \lambda)$, then the characteristics function is defined as;

$$\phi_x(it) = E(e^{itx}) = \int_0^\infty e^{itx} f(x, \alpha, \beta, \lambda) dx$$

$$\phi_x(it) = \int_0^\infty \frac{e^{itx} \lambda^{2\alpha} x^{\alpha-1} e^{-\frac{\lambda^2 x}{\beta}}}{\sqrt{\alpha\beta}^\alpha} dx \tag{42}$$

$$\phi_x(it) = \frac{\lambda^{2\alpha}}{\sqrt{\alpha\beta^\alpha}} \int_0^\infty x^{\alpha-1} e^{itu} \frac{-\lambda^{2x}}{\beta} dx \tag{43}$$

$$\phi_x(it) = \frac{\lambda^{2\alpha}}{\sqrt{\alpha\beta^\alpha}} \int_0^\infty x^{\alpha-1} e^x \left(\frac{it-\lambda^2}{\beta}\right) dx \tag{44}$$

Let $u = x \left(\frac{it-\lambda^2}{\beta}\right)$, $x = \frac{u}{\frac{it-\lambda^2}{\beta}}$, $dx = \frac{du}{\frac{it-\lambda^2}{\beta}}$ (45)

$$\frac{\lambda^{2\alpha}}{\sqrt{\alpha\beta^\alpha}} \int_0^\infty \left(\frac{u}{\frac{it-\lambda^2}{\beta}}\right)^{\alpha-1} e^u \cdot \frac{1}{\frac{it-\lambda^2}{\beta}} du$$

$$\phi_x(it) = \frac{\lambda^{2\alpha}}{\sqrt{\alpha\beta^\alpha}} \int_0^\infty u^{\alpha-1} \cdot \left(\frac{1}{\frac{it-\lambda^2}{\beta}}\right) \cdot \frac{1}{\frac{it-\lambda^2}{\beta}} e^u du \tag{46}$$

$$\phi_x(it) = \frac{\lambda^{2\alpha}}{\sqrt{\alpha\beta^\alpha}} \int_0^\infty \left(\frac{1}{\frac{it-\lambda^2}{\beta}}\right) \cdot u^{\alpha-1} e^u du \tag{47}$$

$$\phi_x(it) = \frac{\lambda^{2\alpha}}{\sqrt{\alpha\beta^\alpha} \left(\frac{it-\lambda^2}{\beta}\right)^\alpha} \int_0^\infty u^{\alpha-1} e^u du \tag{46}$$

$$\phi_x(it) = \frac{\lambda^{2\alpha}}{\sqrt{\alpha\beta^\alpha} \left(\frac{it-\lambda^2}{\beta}\right)^\alpha} \cdot \sqrt{\alpha} \tag{47}$$

$$\phi_x(it) = \frac{\lambda^{2\alpha}}{\beta^\alpha \left(\frac{it-\lambda^2}{\beta}\right)^\alpha} \quad \text{OR} \quad \phi_x(it) = \left[\frac{\lambda^2}{\beta \left(\frac{it-\lambda^2}{\beta}\right)} \right]^\alpha$$

Theorem 4: The cumulative distribution function (CDF) of the Gamma-Exponential-Exponential distribution is given = $F(x) \int_0^\infty f(x) dx$

$$= \int_0^\infty \frac{\lambda^{2\alpha} x^{\alpha-1} e^{-\frac{\lambda^{2x}}{\beta}}}{\sqrt{\alpha\beta^\alpha}} dx \tag{48}$$

$$= \frac{\lambda^{2\alpha}}{\sqrt{\alpha\beta^\alpha}} \int_0^\infty x^{\alpha-1} e^{-\frac{\lambda^{2x}}{\beta}} dx \tag{49}$$

Let $u = \frac{\lambda^{2x}}{\beta}$, $x = \frac{u\beta}{\lambda^2}$, $dx = \frac{\beta du}{\lambda^2}$

$$= \frac{\lambda^{2\alpha}}{\sqrt{\alpha\beta^\alpha}} \int_0^\infty \left(\frac{u\beta}{\lambda^2}\right)^{\alpha-1} e^{-u} \cdot \frac{\beta du}{\lambda^2} \tag{50}$$

$$= \frac{\lambda^{2\alpha}}{\sqrt{\alpha\beta^\alpha}} \int_0^\infty u^{\alpha-1} \beta^{\alpha-1} \beta \cdot \left(\frac{1}{\lambda^2}\right)^{\alpha-1} \left(\frac{1}{\lambda^2}\right) e^{-u} du$$

$$= \frac{\lambda^{2\alpha}}{\sqrt{\alpha\beta^\alpha}} \int_0^\infty \beta^\alpha \frac{1}{(\lambda^2)^\alpha} u^{\alpha-1} e^{-u} du \tag{51}$$

$$= \frac{\lambda^{2\alpha}}{\sqrt{\alpha\beta^\alpha}} \cdot \frac{\beta^\alpha}{\lambda^{2\alpha}} \int_0^\infty u^{\alpha-1} e^{-u} du \tag{52}$$

$$F(x) = \frac{1}{\sqrt{\alpha}} \int_0^\infty u^{\alpha-1} e^{-u} du \tag{53}$$

Skewness and Kurtosis

Skewness quantifies the extent to which a distribution deviates from symmetry, whereas kurtosis measures the degree of peakedness and describes the overall shape of the probability distribution.

$$Sk = \frac{E(x-\mathcal{M})^3}{\delta^3} = \frac{\mathcal{M}^3}{\delta^3} = \frac{\mu_3^i - 3\mu_1^1\mu_2^i + \mu_1^i}{\delta^3}$$

$$= \frac{\frac{\alpha(\alpha+1)(\alpha+2)\beta^3}{\lambda^6}}{\left[\frac{\alpha\beta^2}{\lambda}\right]^3} - 3 \left(\frac{\alpha\beta}{\lambda^2} \left(\alpha(\alpha+1) \frac{\beta^2}{\lambda^4} \right) + 2 \left(\frac{\alpha^3\beta^3}{\lambda^6} \right) \right) \tag{54}$$

$$= \frac{\alpha\beta^3}{\lambda^6} (\alpha+1)(\alpha+2)\beta^3 - \frac{3\alpha^2\beta^2}{\lambda^6} (\alpha+1) + \frac{2\alpha^3\beta^3}{\lambda^6} \div \left[\frac{\beta\sqrt{\alpha}}{\lambda^2} \right]^3$$

$$= \frac{\alpha\beta^3(\alpha+1)(\alpha+2) - 3\alpha^2\beta^3(\alpha+1) + 2\alpha^3\beta^3}{\lambda^6} \times \frac{\lambda^6}{\alpha^{\frac{3}{2}}\beta^2} \tag{55}$$

$$= \frac{\alpha(\alpha+1)(\alpha+2) - 3\alpha^2(\alpha+1) + 2\alpha^2}{\alpha^{\frac{3}{2}}}$$

$$= \frac{\alpha[(\alpha+1)(\alpha+2) - 3\alpha^2(\alpha+1) + 2\alpha^2]}{\alpha^{\frac{3}{2}}} \tag{56}$$

$$= \frac{\alpha^2 + 2\alpha + \alpha + 2 - 3\alpha + 2\alpha^2}{\sqrt{\alpha}}$$

$$= \frac{2}{\sqrt{\alpha}}$$

(57)

Kurtosis

$$K = \frac{E(x-\mu)^4}{\sigma^4} = \frac{\mu^4}{\sigma^4} = \frac{\mu_4^1 - 4\mu_1^1\mu_3^1 + 6\mu_1^2\mu_2^1 - 3\mu_1^4}{\sigma^4}$$

$$K = \frac{\frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)\beta^4}{\lambda^8} - 4 \left[\frac{\alpha\beta}{\lambda^2} \left(\frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)\beta^4}{\lambda^6} \right) \right] + \frac{6\alpha^2\beta^2}{\lambda^4} \left(\frac{\alpha\beta^2}{\lambda^4} (\alpha+1) \right) - \frac{3\alpha^4\beta^4}{\lambda^8}}{\left[\frac{\alpha\beta^2}{\lambda^4} \right]^4} \tag{58}$$

$$K = \frac{\frac{\alpha\beta^4}{\lambda^8} (\alpha+1)(\alpha+2)(\alpha+3) - \frac{4\alpha^2\beta^4}{\lambda^2} (\alpha+1)(\alpha+2) + \frac{6\alpha^2\beta^2}{\lambda^4} \left(\frac{\alpha\beta^2}{\lambda^4} (\alpha+1) \right) - \frac{3\alpha^4\beta^4}{\lambda^8}}{\left[\frac{\alpha\beta^2}{\lambda^4} \right]^4} \tag{59}$$

$$K = \frac{\frac{\alpha\beta^4}{\lambda^8} (\alpha+1)(\alpha+2)(\alpha+3) - \frac{4\alpha\beta}{\lambda^8} (\alpha+1)(\alpha+2) + \frac{6\alpha^3\beta^4}{\lambda^4} (\alpha+1) - \frac{3\alpha^4\beta^4}{\lambda^8}}{\frac{\beta^4\alpha^2}{\lambda^8}} \tag{61}$$

$$K = \frac{\alpha\beta^4(\alpha+1)(\alpha+2)(\alpha+3) - 4\alpha^2\beta^4(\alpha+1)(\alpha+2) + 6\alpha^3\beta^4(\alpha+1) - 3\alpha^4\beta^4}{\lambda^8} \times \frac{\lambda^8}{\alpha^2\beta^4} \tag{62}$$

$$K = \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3) - 4\alpha^2(\alpha+1)(\alpha+2) + 6\alpha^3(\alpha+1) - 3\alpha^4}{\alpha} \tag{63}$$

$$K = \frac{\alpha^3 + 6\alpha^2 + 11\alpha + 6 - 4\alpha^3 - 12\alpha^2 - 8\alpha + 6\alpha^3 + 6\alpha^2 - 3\alpha^2}{\alpha} \tag{64}$$

$$K = \frac{3\alpha + 6}{\alpha}$$

(65)

Method of maximum likelihood

This section focuses on determining the Maximum Likelihood Estimates (MLE) for the parameters of the Gamma- Exponential Exponential Distribution (GEED). The concept behind MLE is to identify the parameter values that maximize the likelihood of observing the sample data (Mood *et al.*, 1974). This process starts with formulating the likelihood function based on the sample data. The likelihood represents the probability of obtaining the specific data set under a chosen probability model, which includes the unknown parameters. The parameter values that maximize this likelihood are referred to as the maximum likelihood estimates (Elgarhy *et al.*, 2017).

$$l(x, \alpha, \lambda) = \left(\frac{\lambda^{2\alpha}}{\sqrt{\alpha\beta^\alpha}}\right)^n \prod_{i=1}^n x_i^{\alpha-1} \exp\left(\frac{-\lambda^2}{\beta} \sum_{i=1}^n x_i\right) \tag{66}$$

Taking the natural log of both sides

$$\log l = 2\alpha \cdot n \cdot \log \lambda - n \log \sqrt{\alpha\beta^\alpha} + (\alpha - 1) \sum_{i=1}^n x_i - \frac{\lambda^2}{\beta} \sum_{i=1}^n x_i \tag{67}$$

Therefore, the MLE which maximizes the above satisfy the following normal equation;

$$\frac{d \log l}{d \lambda} = 2n \log \lambda - n \sqrt{\alpha^1 \beta^2} + \sum_{i=1}^n x_i \tag{68}$$

$$\frac{d \log l}{d \lambda} = \frac{2\alpha n}{\lambda} - \frac{2\lambda}{\beta} \sum_{i=1}^n x_i \tag{69}$$

Simulation study of the Gamma-Exponential Exponential Distribution (GEED)

Torabi (2008) introduced a general method for estimating parameters through spacing known as maximum spacing distance estimator (MSDE). Torabi and Bagheri (2010) and Torabi and Montazeri (2014) used different MSDEs to compare with the MLEs. Here, we compare MLEs to MSDEs “minimum spacing absolute distance estimator” (MSADE) and “minimum spacing absolute-log distance estimator” (MSALDE) of the GEED distribution. For mathematical details, the reader is

referred to Torabi and Bagheri (2010) and Torabi and Montazeri (2014). We simulate the Gamma-Exponential-Exponential Distribution (GEED) for $n = 50, 100, 200, 300$ and 500 with $x = 1.5, \lambda = 0.5, \alpha = 2$ and $\beta = 0.5$. For each sample size, we compute the MLEs, MSADEs and MSALDEs of the parameters. We repeat this process 1,000 times and obtain the average estimates (AEs), biases and mean square error (MSEs). The results are reported in Table 1. We note that the MSEs of MSADEs and MSALDEs are less than the MSEs of MLEs.

Table 1: Estimated AE and MSE of MLE, MSADE and MSALDE of the parameters based on 1000 simulations of the Gamma-Exponential-Exponential distribution for $x = 1.5$, $\lambda = 0.5$, $\alpha = 2$ and $\beta = 0.5$ with $n= 50, 100, 200, 300$ and 500 .

Different method		MLE		MSADE		MSALDE	
n	parameters	A.E	MSE	A.E	MSE	A.E	MSE
50	x	1.778	4.162	1.372	0.181	1.527	0.526
	λ	1.162	1.662	0.966	0.271	0.666	0.171
	α	0.952	0.811	0.572	0.124	0.547	0.162
	β	2.001	5.227	2.028	1.346	2.101	2.272
100	x	1.567	1.292	1.272	0.185	1.378	0.176
	λ	0.571	1.172	0.663	0.410	0.555	0.467
	α	0.515	0.224	0.567	0.096	0.551	0.109
	β	1.653	2.617	1.963	1.614	2.111	1.124
200	x	1.755	0.613	1.582	0.122	1.650	0.356
	λ	0.826	0.334	0.531	0.027	0.506	0.028
	α	0.678	0.202	0.521	0.021	0.494	0.025
	β	2.014	1.343	1.819	0.173	1.888	1.232
300	x	1.624	0.663	1.674	0.151	1.559	0.254
	λ	0.709	0.226	0.547	0.052	0.577	0.087
	α	0.664	0.157	0.520	0.033	0.532	0.056
	β	2.047	1.277	1.939	0.252	1.852	0.962
500	x	1.462	0.267	1.262	0.065	1.555	0.169
	λ	0.530	0.128	0.677	0.062	0.511	0.036
	α	0.528	0.137	0.512	0.012	0.511	0.066
	β	2.156	0.804	1.662	0.155	1.823	0.655

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