Analyzing Patient Flow Dynamics: An M/M/1 Queue Model with Vacations in Hospital Outpatient Services. A Case Study of the Regional Hospital, Bamenda

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ABSTRACT

Globally, healthcare systems face the pervasive challenge of optimizing patient flow, minimizing wait times, and enhancing service delivery in this research work, an M/M/1 queueing system is considered with impatient customers and a variant of multiple vacation policy, where the case that customer impatience is due to the servers' vacation is examined. Whenever a system becomes empty, the server takes a vacation. However, the server is allowed to take a maximum number of vacations, denoted by K vacations, if the system remains empty after the end of a vacation. We derive the probability generating functions of the steady-state probabilities and obtain the closed-form expressions of the system sizes when the server is in different states. In addition, the closed-form expressions for other important performance measures is obtained. Finally, some numerical results are presented. Our result shows that $E(L_K)$ and the mean system size $E(L)$ all decrease with θ for any finite K whereas P_V and P_b neither increase nor decrease with θ when K = 2 and K = 3.

Keywords: M/M/1 queue; Synchronous working vacation; Impatient customers; Generating function.

BACKGROUND STUDY

Globally, healthcare systems face the pervasive challenge of optimizing patient flow, minimizing wait times, and enhancing service delivery (Amina et al, 202; Ankita, 2022). Lengthy waiting times in outpatient services are particularly detrimental, leading to diminished patient satisfaction, potentially compromised clinical outcomes, and a reduction in overall healthcare efficiency and accessibility (Asssia, 2016, Dorac et al, 2022, and Eitan and Uri, 2006).

Healthcare delivery is increasingly challenged by rising patient demand, limited resources, and the need for efficient operations (Litvak, 2010). Long wait times in outpatient services have been consistently linked to patient dissatisfaction, reduced adherence to treatment plans, and negative health outcomes (Bleich, Ozaltin, & Murray, 2009). Long wait times can be attributed to various factors, including systemic inefficiencies, communication breakdowns, and psychological aspects related to waiting (Davis et al., 2018; Smith, 2020). According to Encho et al, 2025, waiting-time, problem arises whenever the demand for customer service cannot perfectly be matched by a set of well-defined service facilities. That is, there is more demand for service than there is facility for service available. This may be adduced to shortage of available services, or limitation to the amount of service that can be provided consequently, the identification and mitigation of factors contributing to these delays are paramount for healthcare administrators and policymakers seeking to improve the quality and effectiveness of care.

Queueing theory offers a powerful and versatile analytical framework for modeling and optimizing patient flow dynamics within healthcare settings (Hanumanthar and Vasanta, 2017; and Mokhitar et al. 2020). The M/M/1 queueing model, a foundational cornerstone of queueing theory, provides a simplified yet insightful representation of a single-server queueing system characterized by Markovian (i.e., exponential) arrival and service processes (Pllabi and Amit, 2015). In this idealized model, patients arrive randomly at an outpatient clinic, potentially forming a queue if the attending doctor or nurse is currently occupied, and are subsequently served based on a first-come, first-served (FIFO) service discipline. The M/M/1 model rests on the assumption that both the inter-arrival times between successive patients and the service times for individual patients follow exponential probability distributions (Rahegh and Bhupender, 2020).

The M/M/1 model serves as an initial analytical tool to demonstrate that substantive waiting times will invariably emerge in outpatient service environments when the patient arrival rate (λ) closely approaches or surpasses the service rate (μ) of the healthcare provider (Shenghl et al, 2024). Specifically, as system utilization (ρ), defined as the ratio of arrival rate to service rate ($\rho = \lambda/\mu$), increases and approaches 1, the average waiting time experienced by patients in the queue (Wq) increases non-linearly and may quickly become excessive (Zahiaet et al, 2023; Kumar and Reghunathan, 2022). This analytical insight underscores the challenges encountered in many outpatient settings, where elevated patient loads coupled with constrained provider capacity create conditions conducive to prolonged waiting periods.

The M/M/1 queueing model, characterized by Poisson arrivals, exponential service times, and a single server, provides a fundamental starting point for analyzing queueing systems (Gross et al., 2018). However, its simplicity often limits its applicability to complex healthcare environments where physician breaks, administrative tasks, and variations in patient needs lead to periods of server unavailability (vacations) (Doshi, 1986).

To better capture the realities of outpatient services, queueing models that incorporate server vacations are essential (Tian & Zhang, 2006). This study proposes to analyze patient flow dynamics at the Regional Hospital Bamenda's outpatient service using an M/M/1 queueing model with vacations, aiming to provide insights into optimizing resource allocation, reducing patient wait times, and improving the overall patient experience.

However, the basic M/M/1 queueing model possesses limitations that restrict its ability to fully capture the complex operational realities of contemporary healthcare systems (Davis and Heineke, 2021). A central limitation arises from the assumption of uninterrupted and continuous service availability. In real-world clinical practice, healthcare providers are not constantly engaged in direct patient care; they routinely take scheduled breaks, participate in meetings or administrative tasks, or are temporarily diverted to handle urgent clinical matters. These periods of provider unavailability effectively introduce interruptions in service, which can be modeled using the concept of "vacations" within queueing theory (Kumar and Veinartz, 2020; Bitnar and Wang,2019).

Queueing models incorporating server vacations represent a significant extension to the basic M/M/1 model by explicitly accounting for periods during which the server is temporarily unavailable for patient service (Chebat and Michan, 2020). These extended models offer a more accurate portrayal of the stochastic dynamics prevalent in healthcare delivery and permit more robust assessment of patient waiting times under varying operational and scheduling scenarios (Kwortnik and Thompson, 2021; Baker and and Cameron, 2021). Further, vacation models provide insights to the best operational procedures, such as non-exhaustive or exhaustive methodologies. Where exhaustive states that the provider should not take a break till all the patients are served, non-exhaustive states that the provider should take breaks.

The Regional Hospital Bamenda, situated in Cameroon, exemplifies a healthcare facility in a developing-world context that contends with significant challenges in managing patient flow and mitigating lengthy waiting times in its outpatient service areas (Roehrich and Grovold, 2019). These challenges are further exacerbated by resource scarcity, chronic staffing shortages, and increasing demand for diverse healthcare services within the community. In order to effectively address the complex clinical and operational challenges, it is important to address specific solutions (Kumar and Raghunathan, 2022).

Therefore, gaining a comprehensive and nuanced understanding of patient flow dynamics within the Regional Hospital Bamenda setting is of paramount importance for the development and implementation of targeted interventions that can effectively improve service delivery processes and concomitantly enhance the overall patient experience (Kumar and Reinartz, 2020). By developing and utilizing queueing-based models, this will contribute in identifying specific recommendations that would reduce congestion (Vehoet et al, 2021; Huang et al, 2022).

Accordingly, this research study intends to overcome the intrinsic limitations of the basic M/M/1 queueing model through the application of an M/M/1 queueing model with vacations specifically tailored to analyzing patient flow characteristics within the outpatient service department of the Regional Hospital Bamenda. The main point that should be considered is to avoid generalizations, so that the operational model can be specifically applied (Kwortnik et al, 2020; Lemon and Verhoef, 2019).

MODEL DESCRIPTION

We consider an *M/M/1* queueing system with impatient customers and a variant of a multiple vacation policy. Customers arrive according to a Poisson process at rate λ. The service is provided by a single server, who serves the customers on a first-come first-served ($FCFS$) basis. The service times follow an exponential distribution with a service rate μ. When the server finishes serving a customer and finds the system empty, the server leaves for a vacation. If the server finds a customer at a vacation completion instant, the server returns to serve customers immediately. Otherwise, the server will take vacations consecutively until the server has taken a maximum number of vacations, denoted by K vacations, and then the server stays idle and waits to serve the next arrival. The vacation times are assumed to be exponentially distributed with vacation rate β. During the vacation, customers become impatient. That is, whenever a customer arrives at the system, it activates an "impatience timer" T, which is exponentially distributed with parameter θ. If the customer's service has not been completed before the customer's timer expires, the customer abandons the queue, never to return.

Stationary Analysis

A differential equation is developed for the probability generating functions of the steady-state probabilities and solved to obtained an expression for the mean system sizes and important performance measure when the server is in different states

Generating Functions

Let $L(t)$ denote the number of customers in the system at time t, and let $J(t)$ denote the status of the server at time t, which is defined as follows:

$J(t) = j$ denotes that the server is taking the $(j+1)th$ vacation at time t for $j = 0, 1, \dots, K-1$, while $J(t) = K$ denotes that the server is idle or busy at time t.

Then, the process $\{(L(t),\ J(t), t \geq 0)\}$ defines a continuous-time Markov process with state space $\Omega = \{(n, j): n \geq 0,\ j = 0, 1, \dots, K\}$.

Let $P_{nj} = \lim t \to \infty\ P\{L(t) = n, J(t) = j\}, n \geq 0, j = 0, 1, \dots, K$, denote the steady-state probabilities of the process $\{(L(t), J(t), t \geq 0)\}$. Then, the set of balance equations is given as follows:

$(\lambda + \beta)P_{00} = \theta p_{10} + \mu P_{1K},$……………………………………………………………(1)

$(\lambda + \beta + n\theta)P_{n0} = \lambda P_{n-10} + (n+1)\theta P_{n+10}, n \geq 0,$ …………………………………… (2)

$(\lambda + \beta)P_{0j} = \theta P_{1j} + \beta P_{0j-1}, j = 1,2, \dots, K-1$ ………………………………………… (3)

$(\lambda + \beta + n\theta)P_{nj} = \lambda P_{n-1j} + (n+1)\theta P_{n+1j},\ j = 1,2, \dots K-1\ \ n \geq 1$………………… .(4)

$\lambda P_{0K} = \beta P_{0K-1}$……………………………………………………………………………(5)

$(\lambda + \mu)P_{nK} = \lambda P_{n-1K} + \mu P_{n+1K} + \beta \sum_{j=0}^{K-1} P_{nj},\ \ \ n \geq 1$…………………………………(6)

And normalizing condition:

$\sum_{n=0}^{\infty}\sum_{j=0}^{K}P_{nj}=1$……………………………………………………………………… 7

Define the probability generating functions $(PGFs)$ as:

$G_j(z)=\sum_{n=0}^{\infty}P_{nj}z^n,\ 0\leq z\leq 1, j=0,1,\ldots,K.$

Define,

$G_j^{'}(z)=\frac{d}{dz}G_j(z),\ j=0,1,\ldots,K.$

Then, multiplying each equation for n in Equations (1), (2), (3) and (4) by z^n, and summing all possible values of n and re-arranging terms, we get;

$\theta(1-z)G_0^{'}(z)-[\lambda(1-z)+\beta]G_0(z)=-\mu P_{1K}$……………………………………8

and

$\theta\ (1-z)G_j^{'}(z)-[\lambda(1-z)+\beta]G_j(z)=-\beta P_{0j-1}\ j=1,2,\ldots,K-1$……………..9

Similarly, using Equations (5) and (6) we obtain:

$(1-z)(\lambda z-\mu)G_K(z)=\beta z\sum_{j=0}^{K-1}G_j(z)+(z-1)\mu P_{0K}-\left[\mu P_{1K}+\beta\sum_{j=0}^{K-2}P_{0j}\right]z$…. (10)

we solve the differential equations (8) and (9) by using the method in Altman and Yechiali (2006).

Solutions of the Differential Equations

Equation (8) can be written as follows by taking $\theta(1-z)$ to divide through the equation:

$\frac{\theta(1-z)G_0^{'}(z)-[\lambda(1-z)+\beta]G_0(z)}{\theta(1-z)}=\frac{-\mu P_{1K}}{\theta(1-z)}$

i.e $G_0^{'}(z)-\left[\frac{\lambda}{\theta}+\frac{\beta}{\theta(1-z)}\right]G_0(z)=-\frac{\mu P_{1K}}{\theta(1-z)}$…………………………………………(11)

In order to solve the differential Equation (11), we obtain the Integration Factor and multiply both sides of Equation (11) by Integrating Factor (I.F.).

i.e. $\text{I.F}=\ e^{\int Pdx}$ from the differential Equation of the form $\frac{dz}{dx}+Pz=Q.$

This implies,

$\text{I.F.}=e^{-\left[\int\left(\frac{\lambda}{\theta}+\frac{\beta}{\theta(1-z)}\right)dz\right]}$

$\left[\int\left(\frac{\lambda}{\theta}+\frac{\beta}{\theta(1-z)}\right)dz\ \right]=\frac{\lambda}{\theta}z+\frac{\beta}{\theta}In\ (1-z)=\frac{\lambda}{\theta}z+In\ (1-z)^{\frac{\beta}{\theta}}=e^{-\frac{\lambda}{xi}z+In(1-z)^{\frac{\beta}{\theta}}}$

$e^{-\frac{\lambda}{\theta}z}.e^{In\ (1-z)^{\frac{\beta}{\theta}}}$

$\therefore I.F.=e^{-\frac{\lambda}{\theta}z}(1-z)^{\frac{\beta}{\theta}}$

Use I.F. to multiply both sides of equation (11)

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$$G_0'(z)e^{-\frac{\lambda}{\theta}}(1-z)^{\frac{\beta}{\theta}} - \left[\frac{\lambda}{\theta} + \frac{\beta}{\theta(1-z)}\right] e^{-\frac{\lambda}{\theta}z}(1-z)^{-\frac{\beta}{\theta}}G_0(z) = -\frac{\mu P_{1K}}{\theta(1-z)} e^{-\frac{\lambda}{\theta}z}(1-z)^{\frac{\beta}{\theta}}$$

$$\frac{d}{dz}[e^{-\frac{\lambda}{\theta}z}(1-z)^{\frac{\beta}{\theta}}G_0(z) = \frac{\mu P_{1K}}{\theta} e^{-\frac{\lambda}{\theta}z}(1-z)^{\frac{\beta}{\theta}-1}$$

By integrating both sides from 0 to z, we have;

$$\int_0^z d(G_0(x)e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}} = -\int_0^z \left(\frac{\mu P_{1K}}{\theta} e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}-1}\right) dx$$

$$\left[G_0(x)e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}}\right]_0^z = -\frac{\mu}{\theta} \int_0^z \left(P_{1K}e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}-1}\right) dx$$

$$\left[G_0(z)e^{-\frac{\lambda}{\theta}z}(1-z)^{\frac{\beta}{\theta}} - G_0(0)e^{-\frac{\lambda}{\theta}0}(1-0)^{\frac{\beta}{\theta}}\right] = \frac{\mu}{\theta}\int_0^z \left(P_{1K}e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}-1}\right) dx$$

$$\left[G_0(z)e^{-\frac{\lambda}{\theta}z}(1-z)^{\frac{\beta}{\theta}} - G_0(0)\right] = \frac{-\mu}{\theta} \int_0^z \left(P_{1K}e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}-1}\right) dx$$

$$G_0(z)e^{-\frac{\lambda}{\theta}z}(1-z)^{\frac{\lambda}{\theta}z}(1-z)^{\frac{\beta}{\theta}} = G_0(0) - \frac{\mu}{\theta}\int_0^z \left(P_{1K}e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}-1}\right) dx$$

$$G_0(z)e^{-\frac{\lambda}{\theta}z} = \left[\frac{G_0(0)-\frac{\mu}{\theta}\int_0^z\left(P_{1K}e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}-1}\right)dx}{(1-z)^{\frac{\gamma}{\xi}}}\right]$$

Therefore,

$$G_0(z) = e^{\frac{\lambda}{\xi}z}\left[\frac{G_0(0)-\frac{\mu}{\theta}\int_0^z\left(P_{1K}e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}-1}\right)dx}{(1-z)^{\frac{\beta}{\theta}}}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (12)$$

Since $G_0(1) = \sum_{n=0}^{\infty} P_{n0} < \infty$ and $z = 1$ is the root of the denominator of the right-hand side of Equation (12), we have that $(z = 1)$ must be the root of the numerator of the right-hand side of Equation (12). So, we obtain:

$$G_0(0) = \frac{\mu}{\theta} CP_{1K} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

$$where\ C = \int_0^1 e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}-1}dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

Noting $G_0(0) = P_{00}$, Equation (54) implies;

$$P_{1K} = \frac{\theta}{\mu C} P_{00} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

Substituting Equation (15) into Equation (12), we obtain,

$$G_0(z) = \frac{e^{\frac{\lambda}{\theta}z}}{(1-z)^{\frac{\beta}{\xi}}}\left[G_0(0) - \frac{\mu}{\theta}\left(\frac{\theta}{\mu C}P_{00}\right)\int_0^z(1-x)^{\frac{\beta}{\theta}-1}e^{-\frac{\lambda}{\theta}x}dx\right]$$

$$G_0(z) = \frac{e^{\frac{\lambda}{\theta}z}}{(1-z)^{\frac{\beta}{\theta}}}\left[1 - \frac{1}{C}\int_0^z(1-x)^{\frac{\beta}{\theta}-1}e^{-\frac{\lambda}{V}x}\,dx\right]P_{00} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

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Equation (9) can also be written as:

$$\frac{\theta(1-z)G_j'(z)-[\lambda(1-z)+\beta]G_j(z)=-\beta P_{0j-1}}{\theta(1-z)}$$

$$G_j'(z)-\left[\frac{\lambda}{\theta}+\frac{\beta}{\theta(1-z)}\right]G_j(z)=-\frac{\beta P_{0j-1}}{\theta(1-z)} \quad \text{……………………………………………… (17)}$$

In order to solve the differential Equation (17), we obtain the Integration Factor and multiply both sides of Equation (17) by Integrating Factor (I.F.).

i.e $I.F.=e^{\int Pdx}$ from the differential Equation of the form $\frac{dz}{dx}+Pz=Q$.

This implies;

$$I.F.=e^{-\left[\int\left(\frac{\lambda}{\theta}+\frac{\beta}{\theta(1-z)}dz\right)\right]}$$

i.e. $\left[\int\left(\frac{\lambda}{\theta}+\frac{\beta}{\theta(1-z)}\right)dz\right]=\frac{\lambda}{\theta}z+\frac{\beta}{\theta}In(1-z)=\frac{\lambda}{\theta}z+In\,(1-z)^{\frac{\beta}{\theta}}=e^{-\frac{\lambda}{\theta}z+In(1-z)^{\frac{\beta}{\theta}}}$

$$e^{-\frac{\lambda}{\theta}z}.e\hat{}(In\,(1-z)^{\frac{\beta}{\theta}}$$

$$\therefore I.F.=e^{-\frac{\lambda}{\theta}z}(1-z)^{\frac{\beta}{\theta}}$$

Use I.F. to multiply both sides of equation (17)

$$G_j^{'}(z)e^{-\frac{\lambda}{\theta}z}(1-z)^{\frac{\beta}{\theta}}-\left[\frac{\lambda}{\theta}+\frac{\beta}{\theta(1-z)}\right]\,e^{-\frac{\lambda}{\theta}z}(1-z)^{\frac{\beta}{\theta}}G_j(z)=-\frac{\beta P_{0j-1}}{\xi(1-z)}e^{-\frac{\lambda}{\xi}z}(1-z)^{\frac{\beta}{\xi}}$$

$$\frac{d}{dz}\left[e^{-\frac{\lambda}{\theta}z}(1-z)^{\frac{\beta}{\theta}}G_j(z)\right]=-\frac{\beta P_{0j-1}}{\theta}e^{-\frac{\lambda}{\theta}z}(1-z)^{\frac{\beta}{\theta}-1}$$

By integrating both sides from 0 to z, we have

$$\int_0^z d\left(G_j(x)e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}}\right)=\int_0^z\frac{\beta P_{0j-1}}{\theta}e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}-1}dx$$

$$\left[G_j(x)e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}}\right]_0^z=\frac{\beta}{\xi}\int_0^z(P_{0j-1}e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}-1}dx$$

$$\left[G_j(z)e^{-\frac{\lambda}{\theta}z}(1-z)^{\frac{\beta}{\theta}}-G_j(0)e^{-\frac{\lambda}{\theta}0}(1-0)^{\frac{\beta}{\theta}}\right]=-\frac{\beta}{\theta}\int_0^z(P_{0j-1}e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}-1}\,dx$$

$$\left[G_j(z)e^{-\frac{\lambda}{\theta}z}(1-z)^{\frac{\beta}{\theta}}-G_j(0)\right]=-\frac{\beta}{\theta}\int_0^z(P_{0j-1}e^{-\frac{\lambda}{\theta}x}\,(1-x)^{\frac{\beta}{\theta}-1}dx$$

$$G_j(z)e^{-\frac{\lambda}{\theta}z}(1-z)^{\frac{\beta}{\theta}}=G_j(0)-\frac{\beta}{\theta}\int_0^z(P_{0j-1}e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}-1}dx$$

$$G_j(z)e^{-\frac{\lambda}{\theta}z}=\left[\frac{G_j(0)-\frac{\beta}{\theta}\int_0^z(P_{0j-1}e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}-1}dx}{(1-z)^{\frac{\beta}{\theta}}}\right]$$

Therefore,

$$G_j(z) = e^{\frac{\lambda}{\theta}z}\left[\frac{G_j(0)-\frac{\beta}{\theta}P_{0j-1}\int_0^z\left(e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}-1}\right)dx}{(1-z)^{\frac{\beta}{\theta}}}\right] \quad j = 1,2,\dots,K-1 \text{ .... } (18)$$

Since $G_j(1) = \sum_{n=0}^{\infty} P_{nj} < \infty$ and $z = 1$ is the root of the denominator of the right-hand side of Equation (18), we have that z = 1 must be the root of the numerator of the right-hand side of Equation (18). So, we obtain;

$$P_{0j} = G_j(0) = \frac{\beta}{\theta} C P_{0j-1} \quad j = 1,2,\dots,K-1 \text{ ....}(19)$$

Note,

$$P_{oj-1} = \frac{\theta P_{0j}}{\beta C}$$

Where C is defined by Equation (14), i.e,

$$C = \int_0^1 e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}-1}\, dx$$

Using Equation (19) repeatedly, we obtain;

$$P_{0j} = A^j P_{00}, \ j = 1,2,\dots,k-1 \text{ ....}(20)$$

Where $A = \frac{\beta}{\theta}C$. Substituting Equation (20) into Equation (18), we obtain;

$$G_j(z) = \frac{e^{\frac{\lambda}{\theta}z}}{(1-z)^{\frac{\beta}{\theta}}}[G_j(0) - \frac{\beta}{\theta}\left(\frac{\theta}{\beta C}P_{0j}\right) \int_0^z (1-x)^{\frac{\beta}{\theta}-1} e^{-\frac{\lambda}{\theta}x}\, dx$$

$$G_j(z) = \frac{e^{\frac{\lambda}{\theta}z}}{(1-z)^{\frac{\beta}{\theta}}}\left[P_{0j} - \frac{\beta}{\theta}\left(\frac{\theta}{\beta C}P_{0j}\right) \int_0^z (1-x)^{\frac{\beta}{\theta}-1} e^{-\frac{\lambda}{\theta}x}\, dx\right]$$

$$G_j(z) = \frac{e^{\frac{\lambda}{\theta}z}}{(1-z)^{\frac{\beta}{\theta}}}\left[A^j P_{00} - \frac{\beta}{\theta}\left(\frac{\theta}{\beta C}A^j P_{00}\right) \int_0^z (1-x)^{\frac{\beta}{\theta}-1} e^{-\frac{\lambda}{\theta}x} dx\right]$$

$$G_j(z) = \frac{e^{\frac{\lambda}{\theta}z}A^j}{(1-z)^{\frac{\beta}{\theta}}}\left[1 - \frac{1}{C}\int_0^z (1-x)^{\frac{\beta}{\theta}-1}\, e^{-\frac{\lambda}{\theta}x} dx\ \right] P_{00} \quad j = 1,2,\dots,K-1 \text{ .... } (21)$$

Using Equation (15) and (21), we obtain,

$$\lambda P_{0K} = \beta P_{0K-1}$$

$$P_{0K} = \frac{\beta}{\lambda}A^{K-1}P_{00} \text{ .... } (22)$$

We derive the probability P_{00} and the mean system sizes when the server is in different states.

Mean System Sizes

We derive the probability P_{00} and the mean system sizes when the server is in different states

For $j = 0,1,2,\dots,K$, let L_j be the system size when the server is in the state j. Then, $E(L_j)$ is the mean system size when the server is in the state j, which is defined by:

$E(L_j) = G_j'(1) = \sum_{n=1}^{\infty} nP_{nj}, \;\; j = 0,1,2, \dots, K$

Recall that,

$G_j(z) = \sum_{n=0}^{\infty} P_{nj} z^n$

$G_j^{'}(z) = \sum_{n=1}^{\infty} nP_{nj} z^n \;\; but \;\; z = 1$

$G_j^{'}(1) = \sum_{n=1}^{\infty} nP_{nj}, \;\; j = 0,1,2, \dots, K$

That is, for $j = 0,1,2,\dots,K-1, E(L_j)$ represents the mean system size when the server is taking the $(j+1)th$ vacation, and $E(L_K)$ represents the mean system size when the server is busy or idle. We first derive $E(L_j)$ for $j = 0,1,2,\dots,K-1$.

Making $G_0^{'}(z)$ subject of the formula from equation (11), we have as follow;

$G_0^{'}(z) - \left[\frac{\lambda}{\theta} + \frac{\beta}{\theta(1-z)}\right] G_0(z) = -\frac{\mu P_{1K}}{\theta(1-z)}$

$G_0^{'}(z) = \left[\frac{\lambda G_0(z)}{\theta} + \frac{\beta G_0(z)}{\theta(1-z)} - \frac{\mu P_{1K}}{\theta(1-z)}\right]$

$G_0^{'}(z) = \left[\frac{\lambda(1-z)G_0(z) + \beta G_0(z) - \mu P_{1K}}{\theta(1-z)}\right]$

Taking the limit of both sides as $z \to 1$, then,

$G_0^{'}(1) = \lim_{z\to 1} \frac{\lambda(1-z)G_0(z) + \beta G_0(z) - \mu P_{1K}}{\theta(1-z)}$

Using L'Hopital rule, i.e. differentiating with respect to z the numerator and denominator of $R.H.S$ of $G_0^{'}(1)$ we get:

$G_0^{'}(1) = \lim_{z\to 1} \frac{[\lambda(1-z)+\beta]G_0(z) - \mu P_{1K}}{\theta(1-z)}$

$G_0^{'}(1) = \frac{-\lambda G_0(z) + \beta G_0^{'}(z)}{-\theta}$

$G_0^{'}(1) = \frac{-\lambda G_0(1) + \beta G_0^{'}(1)}{-\theta}$

$-\theta G_0^{'}(1) = -\lambda G_0(1) + \beta G_0^{'}(1)$

$\lambda G_0(1) = \beta G_0^{'}(1) + \theta G_0^{'}(1)$

Thus, we obtain;

$(\beta + \theta) G_0^{'}(1) = \lambda G_0(1)$………………………………………………………………… (23)

Similarly, from Equation (17),

$G_j^{'}(z) - \left[\frac{\lambda}{\theta} + \frac{\beta}{\theta(1-z)}\right] G_j(z) = -\frac{\beta P_{oj-1}}{\theta(1-z)}$

Making $G_j^{'}(z)$ subject of the formula, we have;

$G_j^{'}(z) = \left[\frac{\lambda G_j(z)}{\theta} + \frac{\beta G_j(z)}{\theta(1-z)} - \frac{\mu P_{0j-1}}{\theta(1-z)}\right]$

$$G_j^{'}(z) = \left[\frac{\lambda(1-z)G_j(z)+\beta G_j(z)-\mu P_{0j-1}}{\theta(1-z)}\right]$$

Taking the limit both sides as $z \to 1$, then,

$$G_j^{'}(1) = \lim_{z\to 1}\frac{\lambda(1-z)G_j(z)+\beta G_j(z)-\mu P_{0j-1}}{\theta(1-z)}$$

Using L'Hopital rule, i.e. differentiating with respect to z the numerator and denominator of R.H.S. of $G_j^{'}(1)$ we get:

$$G_j^{'}(1) = \lim_{z\to 1}\frac{[\lambda(1-z)+\beta]G_j(z)-\mu P_{0j-1}}{\theta(1-z)}$$

$$G_j^{'}(1) = \frac{-\lambda G_j(z)+\beta G_j^{'}(z)}{-\theta}$$

$$G_j^{'}(1) = \frac{-\lambda G_j(1)+\beta G_j^{'}(1)}{-\theta}$$

$$-\xi G_j^{'}(1) = -\lambda G_j(1) + \beta G_j^{'}(1)$$

$$\lambda G_j(1) = \beta G_j^{'}(1) + \theta G_j^{'}(1)$$

Thus, we have;

$$(\lambda+\theta)G_j^{'}(1) = \lambda G_j(1),\ \ j = 0,1,2,\dots,K-1 \dots\dots\dots\dots(24)$$

$$G_j^{'}(1) = \left(\frac{\lambda}{\beta+\theta}\right)G_j(1)$$

In this way, equations (23) and (24) imply,

$$E(L_j) = G_j^{'}(1) = \frac{\lambda}{\beta+\theta}G_j(1),\ j = 0,1,\dots,K-1 \dots\dots\dots\dots (25)$$

For $j = 0,1,\dots,K-1$, let $P_{.j} = G_j(1) = \sum_{n=0}^{\infty} P_{jn}$. Then, for $j = 0,1,\dots,K-1, P_{.j}$ represents the probability that the server is taking the $(j+1)th$ vacation, and $P_{.K}$ represents the probability the server is busy or idle.

From Equations (16) and Equation (21), using L'Hopital rule, we get the following: Solving (16) first, i.e.

$$G_0(z) = \frac{e^{\frac{\lambda}{\theta}z}}{(1-z)^{\frac{\beta}{\theta}}}\left[1 - \frac{\int_0^z (1-x)^{\frac{\beta}{\theta}-1}e^{-\frac{\lambda}{\theta}x}dx}{C}\right]P_{00}$$

Let the Integral function in the numerator be represented as I i.e.

$$I = \int_0^z (1-x)^{\frac{\beta}{\theta}-1}e^{-\frac{-\beta}{\theta}x}\,dx$$

Using Integration by art:

$$\int udv = uv - \int vdu$$

$$u = e^{-\frac{\lambda}{\theta}x}$$

$$du = -\frac{\lambda}{\theta}e^{-\frac{\lambda}{\theta}x}$$

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$dv = (1-x)^{\frac{\beta}{\theta}-1}$

$\int dv = \int (1-x)^{\frac{\beta}{\theta}-1}$

$v = -\frac{\theta}{\gamma}(1-x)^{\frac{\beta}{\theta}}$

$I = uv - \int vdu \ = e^{-\frac{\lambda}{\theta}x}\left(-\frac{\theta}{\beta}(1-x)^{\frac{\beta}{\theta}}\right) - \frac{\beta\theta}{\theta\beta}\int_0^z (1-x)^{\frac{\beta}{\theta}} e^{-\frac{\lambda}{\theta}x}\, dx$

$I = -\frac{\theta}{\beta} e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}} - \frac{\lambda}{\theta}\int_0^z (1-x)^{\frac{\beta}{\theta}} e^{-\frac{\beta}{\theta}x}\, dx$

Also, let, $u = (1-x)^{\frac{\beta}{\theta}}$

$du = -\frac{\beta}{\theta}(1-x)^{\frac{\beta}{\theta}-1}dx$

$du = e^{-\frac{\lambda}{\theta}x}dx$

$\int dv = \int e^{-\frac{\lambda}{\theta}x}\, dx$

$v = -\frac{\lambda}{\theta} e^{-\frac{\lambda}{\theta}x}$

$\therefore I = uv - \int vdu$

$= (-(1-x)^{\frac{\beta}{\theta}}\ \frac{\lambda}{\theta} e^{-\frac{\lambda}{\theta}x} - \int -\frac{\lambda}{\theta} e^{-\frac{\lambda}{\theta}x} - \frac{\beta}{\theta}(1-x)^{\frac{\beta}{\theta}-1}dx$

$= -\frac{\gamma}{\theta}(1-x)^{\frac{\beta}{\theta}} e^{-\frac{\beta}{\theta}x} - \frac{\lambda\beta}{\theta^2}\int e^{-\frac{\beta}{\theta}x}(1-x)^{\frac{\beta}{\theta}-1}dx$

Hence;

$I = -\frac{\theta}{\beta} e^{-\frac{\beta}{\theta}x}(1-x)^{\frac{\beta}{\theta}} - \frac{\lambda}{\beta}\left[-\frac{\lambda}{\theta}(1-x)^{\frac{\beta}{xi}} e^{-\frac{\lambda}{\theta}x} - \frac{\lambda\gamma}{\theta^2}\int e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}-1}\, dx\right]$

$I = -\frac{\theta}{\beta} e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}} - \frac{\lambda}{\beta}\left[-\frac{\lambda}{\theta}(1-x)^{\frac{\beta}{\theta}} e^{-\frac{\lambda}{\theta}x} - \frac{\lambda\beta}{\theta^2} I\right]$

$I = -\frac{\theta}{\beta} e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}} + \frac{\lambda^2}{\beta\theta}(1-x)^{\frac{\beta}{\theta}} e^{-\frac{\lambda}{\theta}x} + \frac{\lambda^2}{\theta^2} I$

$I - \frac{\lambda^2}{\theta^2} I = \frac{\lambda^2}{\beta\theta}(1-x)^{\frac{\beta}{\theta}} e^{-\frac{\lambda}{\theta}x} - \frac{\theta}{\beta} e^{-\frac{\lambda}{\theta}x}(1-x)$

$I\left(1-\frac{\lambda^2}{\theta^2}\right) = \frac{\lambda^2}{\beta\theta}(1-x)^{\frac{\beta}{\theta}}\ e^{\frac{\beta}{\theta}x} - \frac{\xi}{\beta} e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}}$

$I\left(\frac{\theta^2-\lambda^2}{\theta^2}\right) = \frac{\lambda^2}{\beta\theta}(1-x)^{\frac{\beta}{\theta}x} e^{-\frac{\lambda}{\xi}x} - \frac{\xi}{\beta} e^{-\frac{\lambda}{\xi}x}(1-x)^{\frac{\beta}{\theta}}$

$I(\theta^2-\lambda^2) = \frac{\lambda^2\theta}{\beta}(1-x)^{\frac{\beta}{\theta}} e^{-\frac{\lambda}{\theta}x} - \frac{\theta^3}{\beta} e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\gamma}{\theta}}$

$$I = \frac{\lambda^2\theta}{\beta(\theta^2-\lambda^2)}(1-x)^{\frac{\beta}{\theta}}e^{-\frac{\lambda}{\theta}x} - \frac{\theta^3}{\beta(\theta^2-\lambda^2)}e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}}$$

$$I = \frac{e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}}}{\beta(\xi^2-\lambda^2)}[\lambda^2\theta - \theta^3]$$

$$I = \frac{\theta}{\beta}e^{-\frac{\beta}{\theta}x}(1-x)^{\frac{\beta}{\theta}}\left[\frac{\lambda^2-\theta^2}{\theta^2-\lambda^2}\right]$$

Taking the definite integral from 0 to z

$$I = \left[\frac{\theta}{\beta}e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}}\left(\frac{\lambda^2-\theta^2}{\theta^2-\lambda^2}\right)\right]_0^z$$

$$I = \frac{\theta}{\beta}e^{-\frac{\lambda}{\theta}z}(1-z)^{\frac{\beta}{\theta}}\left(\frac{\lambda^2-\theta^2}{\theta^2-\lambda^2}\right) - \frac{\beta}{\theta}\left(\frac{\lambda^2-\theta^2}{\theta^2-\lambda^2}\right)$$

$$I = \frac{\theta}{\beta}\left(\frac{\lambda^2-\theta^2}{\theta^2-\lambda^2}\right)\left[e^{-\frac{\lambda}{\theta}z}(1-z)^{\frac{\beta}{\theta}-1}\right]$$

Where, $C = \int_0^1 (1-x)^{\frac{\beta}{\theta}-1}e^{-\frac{\lambda}{\theta}x}dx$

$$C = \left[\frac{\theta}{\beta}e^{-\frac{\lambda}{\theta}x}(1-x)^{\frac{\beta}{\theta}}\left(\frac{\lambda^2-\theta^2}{\theta^2-\lambda^2}\right)\right]_0^1$$

$$C = \left[\frac{\theta}{\beta}e^{-\frac{\lambda}{\theta}1}(1-1)^{\frac{\beta}{\theta}}\left(\frac{\lambda^2-\theta^2}{\theta^2-\lambda^2}\right) - \frac{\theta}{\beta}e^{-\frac{\lambda}{\theta}0}(1-0)^{\frac{\beta}{\theta}}\left(\frac{\lambda^2-\theta^2}{\theta^2-\lambda^2}\right)\right]$$

$$C = -\frac{\theta}{\beta}\left(\frac{\lambda^2-\theta^2}{\theta^2-\lambda^2}\right)$$

Therefore, putting C and I into $G_0(z)$ we have:

$$G_0(z) = \frac{e^{\frac{\lambda}{\theta}z}}{(1-z)^{\frac{\beta}{\theta}}}\left[1 - \frac{\int_0^z (1-x)^{\frac{\beta}{\theta}-1}e^{-\frac{\lambda}{\theta}x}dx}{-\frac{\theta}{\beta}\left(\frac{\lambda^2-\theta^2}{\theta^2-\lambda^2}\right)}\right]P_{00}$$

$$G_0(z) = \frac{e^{\frac{\lambda}{\theta}z}}{(1-z)^{\frac{\beta}{\theta}}}\left[1 + \frac{\int_0^z (1-x)^{\frac{\beta}{\theta}-1}e^{-\frac{\lambda}{\theta}x}dx}{\frac{\theta}{\beta}\left(\frac{\lambda^2-\theta^2}{\theta^2-\lambda^2}\right)}\right]P_{00}$$

Since, $I = \int_0^z (1-x)^{\frac{\beta}{\theta}-1}e^{-\frac{\lambda}{\theta}x}dx = \frac{\theta}{\beta}\left(\frac{\lambda^2-\theta^2}{\theta^2-\lambda^2}\right)\left[e^{-\frac{\lambda}{\theta}z}(1-z)^{\frac{\beta}{\theta}} - 1\right]$

Then, $G_0(z) = \frac{e^{\frac{\lambda}{\theta}z}}{(1-z)^{\frac{\beta}{\theta}}}\left[1 + \frac{\frac{\theta}{\beta}\left(\frac{\lambda^2-\theta^2}{\theta^2-\lambda^2}\right)\left[e^{-\frac{\lambda}{\theta}z}(1-z)^{\frac{\beta}{\theta}-1}\right]}{\frac{\theta}{\beta}\left(\frac{\lambda^2-\theta^2}{\theta^2-\lambda^2}\right)}\right]P_{00}$

$$G_0(z) = \frac{e^{\frac{\lambda}{\theta}z}}{(1-z)^{\frac{\beta}{\theta}}}\left[1 + e^{-\frac{\lambda}{\theta}z}(1-z)^{\frac{\beta}{\theta}} - 1\right]P_{00}$$

$$G_0(z) = \frac{e^{\frac{\lambda}{\theta}z}}{(1-z)^{\frac{\beta}{\theta}}}\left[e^{-\frac{\lambda}{\theta}z}(1-z)^{\frac{\beta}{\theta}}\right]P_{00}$$

$$G_0(z) = \left[\frac{e^{\frac{\lambda}{\theta}z}}{e^{\frac{\lambda}{\theta}z}}\right] P_{00}$$

Using L'hopital rules; $G_0(z) = \lim_{z \to 1} \left[\frac{e^{\frac{\lambda}{\theta}z}}{e^{\frac{\lambda}{\theta}z}}\right] P_{00}$

$$G_0(1) = \frac{\frac{\lambda}{\theta}}{\frac{\lambda}{\theta}} \left[\frac{e^{\frac{\lambda}{\theta}z}}{e^{\frac{\lambda}{\theta}z}}\right] P_{00}$$

$$G_0(1) = \frac{\frac{\lambda}{\xi}}{\frac{\lambda}{\xi}} \left[\frac{e^{\frac{\lambda}{\theta}}}{e^{\frac{\lambda}{xi}}}\right] P_{00}$$

Solving Equation (21) in a similar manner using L'Hopital rules, thus:

$$G_j(z) = \frac{e^{\frac{\lambda}{\theta}z} A^j}{(1-z)^{\frac{\beta}{\theta}}} \left[1 - \frac{1}{C} \int_0^z (1-x)^{\frac{\beta}{\theta}-1} e^{-\frac{\lambda}{\theta}x} dx\right] P_{00} \;\; j = 1,2,\dots,K-1.$$

Since, $I = \int_0^z (1-x)^{\frac{\beta}{\theta}-1} e^{-\frac{\lambda}{\theta}x} dx = \frac{\theta}{\beta}\left(\frac{\lambda^2-\xi^2}{\theta^2-\lambda^2}\right)\left[e^{-\frac{\lambda}{\theta}z}(1-z)^{\frac{\beta}{\theta}} - 1\right]$ and

$$C = -\frac{\theta}{\beta}\left(\frac{\lambda^2-\theta^2}{\theta^2-\lambda^2}\right)$$

Therefore, putting C and I into $G_0(z)$ we have:

$$G_j(z) = \frac{e^{\frac{\lambda}{\theta}z} A^j}{(1-z)^{\frac{\beta}{\theta}}} \left[1 - \frac{\int_0^z (1-z)^{\frac{\beta}{\theta}-1} e^{-\frac{\lambda}{\theta}x} dx}{-\frac{\theta}{\beta}\left(\frac{\lambda^2-\theta^2}{\theta^2-\lambda^2}\right)}\right] P_{00}$$

$$G_j(z) = \frac{e^{\frac{\lambda}{\theta}z} A^j}{(1-z)^{\frac{\beta}{\theta}}} \left[1 + \frac{\int_0^z (1-x)^{\frac{\beta}{\theta}-1} e^{-\frac{\lambda}{\theta}x} dx}{\frac{\theta}{\beta}\left(\frac{\lambda^2-\theta^2}{\xi^2-\lambda^2}\right)}\right] P_{00}$$

$$G_j(z) = \frac{e^{\frac{\lambda}{\theta}z} A^j}{(1-z)^{\frac{\beta}{\theta}}} \left[1 + \frac{\frac{\theta}{\beta}\left(\frac{\lambda^2-\theta^2}{\theta^2-\lambda^2}\right)\left[e^{-\frac{\lambda}{\theta}z}(1-z)^{\frac{\beta}{\theta}} - 1\right]}{\frac{\theta}{\beta}\left(\frac{\lambda^2-\theta^2}{\theta^2-\lambda^2}\right)}\right] P_{00}$$

$$G_j(z) = \frac{e^{\frac{\lambda}{\theta}z} A^j}{(1-z)^{\frac{\beta}{\theta}}} \left[1 + e^{-\frac{\lambda}{\theta}z}(1-z)^{\frac{\beta}{\theta}} - 1\right] P_{00}$$

$$G_j(z) = \frac{e^{\frac{\lambda}{\theta}z} A^j}{(1-z)^{\frac{\beta}{\theta}}} \left[e^{-\frac{\lambda}{\theta}z}(1-z)^{\frac{\beta}{\theta}}\right] P_{00}$$

$$G_j(z) = \left[\frac{e^{\frac{\lambda}{\theta}z} A^j}{e^{\frac{\lambda}{\theta}z}}\right] P_{00}$$

Using L'Hopital rules, $G_j(z) = \lim_{z \to 1} \left[\frac{e^{\frac{\lambda}{\theta}z} A^j}{e^{\frac{\lambda}{\theta}z}}\right] P_{00}$

$$G_j(1) = \frac{\frac{\lambda}{\xi}}{\frac{\lambda}{\xi}} \left[\frac{e^{\frac{\lambda}{\theta}z} A^j}{e^{\frac{\lambda}{\theta}z}}\right] P_{00}$$

$$G_j(1) = \frac{\frac{\lambda}{\xi}}{\frac{\lambda}{\xi}}\left[\frac{e^{\frac{\lambda}{\theta}}A^j}{e^{\frac{\lambda}{\theta}}}\right]P_{00}$$

$$G_j(1) = A^j P_{00}$$

$$\therefore P_{.j} = G_j(1) = A^{j-1}P_{00}\ldots\ldots(26)$$

Using equation (23), Equation (25) can be written as:

$$E(L_j) = \frac{\lambda}{\beta+\theta}G_j(1)\ 0,1,\ldots,K-1$$

$$E(L_j) = \frac{\lambda}{\beta+\theta}A^{j-1}P_{00}\ j = 0,1,\ldots,K-1\ldots\ldots(27)$$

Note that, **f**rom Equation (27) and $0 < A < 1$, it is easy to see that the mean system size $E(L_j)$ is a decreasing convex function of j for $j = 0, 1, \ldots, K - 1$. Furthermore, the mean system size when the server is on vacation, denoted by $E(L_V)$, is obtained as follows:

$$E(L_V) = \sum_{j=0}^{K-1} E(L_j) = \frac{\lambda}{\beta+\theta}.\frac{1-A^K}{A(1-A)}P_{00}\ \ldots\ldots(28)$$

Deriving $P_{.K}$ and P_{00} from Equation (15), (20) and (26), we have

$\mu P_{1K} = \beta P_{.0}$ and $P_{0j-1} = P_{.j},\ j = 1,2,\ldots,K-1.$

Thus, we have:

$$\mu P_{1K} + \beta\sum_{j=0}^{K-2}P_{0j} = \beta\sum_{j=0}^{K-1}P_{.j}\ \ldots\ldots(29)$$

Using equation (29), Equation 11(51) can be written as, i.e.

$$(1-z)(\lambda z-\mu)(G_K(z) = \beta z\sum_{j=0}^{K-1}G_j(z) + (z-1)\mu P_{0K} - \left[\mu P_{1K} + \beta\sum_{j=0}^{K-2}P_{0j}\right]z\ \ (z-1)(\lambda z-\mu)G_K(z) = \beta z\sum_{j=0}^{K-1}G_j(z) + (z-1)\mu P_{0K} - \left[\beta\sum_{j=0}^{K-1}P_{.j}\right]z$$

$$(1-z)(\lambda z-\mu)G_K(z) = \beta z\sum_{j=0}^{K-1}G_j(z) + (z-1)\mu P_{0K} - \beta z\sum_{j=0}^{K-1}P_{.j}$$

$$(1-z)(\lambda z-\mu)G_K(z) = \beta z\sum_{j=0}^{K-1}\left[G_j(z) - P_{.j}\right] + (z-1)\mu P_{0K}$$

$$(1-z)(\lambda z-\mu)G_K(z) = \beta z\sum_{j=0}^{K-1}\left[G_j(z) - P_{.j}\right] - (1-z)\mu P_{0K}$$

$$G_K(z) = \frac{\beta z}{(\lambda z-\mu)}.\frac{\sum_{j=0}^{K-1}\left[G_j(z)-P_{.j}\right]}{(1-z)} - \frac{\mu P_{0K}}{(\lambda z-\mu)}\ldots\ldots(30)$$

Applying L'Hopital rule, thus:

$$G_k(z) = \lim_{z\to 1}\frac{\beta z}{(\lambda z-\mu)}.\lim_{z\to 1}\frac{\sum_{j=0}^{K-1}\left[G_j(z)-P_{.j}\right]}{(1-z)} - \lim_{z\to 1}\frac{\mu P_{0K}}{(\lambda z-\mu)}$$

$$G_K(z) = \frac{\gamma}{(\lambda-\mu)}.\frac{\sum_{j=0}^{K-1}\left[G_j'(z)-P_{.j}\right]}{(-1)} - \frac{\mu P_{0K}}{(\lambda-\mu)}$$

$$G_K(z) = \frac{\beta\sum_{j=0}^{K-1}\left[G_j'(z)-P_{.j}\right]}{(\mu-\lambda)} - \frac{\mu P_{0K}}{(\lambda-\mu)}$$

$G_K(z) = \frac{\beta \sum_{j=0}^{K-1}\left[G_j'(z)-P_{.j}\right]}{(\mu-\lambda)} + \frac{\mu P_{0K}}{(\mu-\lambda)}$

We have;

$$G_K(1) = \frac{\beta \sum_{j=0}^{K-1}\left[G_j'(1)-P_{.j}\right]+\mu P_{0K}}{(\mu-\lambda)} \quad \text{……(31)}$$

Noting $G_K(1) = P_{.K}$ and $G_j'(1) = E(L_j),\ j = 0,1,\dots,K-1$, from equation (31), we obtain:

$$P_{.K} = \frac{\beta \sum_{j=0}^{K-1}\left[E(L_j)-P_{.j}\right]+\mu P_{0K}}{(\mu-\lambda)} \quad \text{……(32)}$$

Substituting equations (22) and (28) into equation (32), i.e.

$P_{0K} = \frac{\beta}{\lambda} A^{K-1} P_{00}$

$E(L_V) = \sum_{j=0}^{K-1} E(L_j) = \frac{\lambda}{\beta+\theta} \cdot \frac{1-A^K}{A(1-A)} P_{00}$

$$P_{.K} = \frac{\beta}{(\mu-\lambda)}\left[\frac{\lambda}{\beta+\theta} \cdot \frac{1-A^K}{A(1-A)} + \frac{\gamma}{\lambda} A^{K-1}\right] P_{00} \quad \text{……(33)}$$

Using the definition of $P_{.j}$, we see that normalizing condition (7) can also be written as:

$$\sum_{j=0}^{K} P_{.j} = 1 \quad \text{…… (34)}$$

Substituting Equation (16) and (33) into Equation (34), thus:

$P_{.j} = G_j(1) = A^{j-1} P_{00}$

$P_{.K} = \frac{\beta}{\mu-\lambda}\left[\frac{\lambda}{\beta+\theta} \cdot \frac{1-A^K}{A(1-A)} + \frac{\beta}{\lambda} A^{K-1}\right] P_{00}$

We have as follows:

$$P_{00} = \left[\frac{\beta\lambda+(\mu-\lambda)\theta}{(\mu-\lambda)(\beta+\theta)} \cdot \frac{1-A^K}{A(1-A)} + \frac{\beta\mu}{\lambda(\mu-\lambda)} A^{K-1}\right]^{-1} \quad \text{…… (35)}$$

Note that, from Equation (32), the inequality $P_{.K} > 0$ is equivalent to $\lambda < \mu$. So $\lambda < \mu$ is a necessary condition for the stability of our system. Therefore, we assume thereafter that $\lambda < \mu$. Substituting Equation (35) into Equation (18), we obtain:

$P_{00} = \left[\frac{\beta\lambda+(\mu-\lambda)\theta}{(\mu-\lambda)(\beta+\theta)} \cdot \frac{1-A^K}{A(1-A)} + \frac{\beta\mu}{\lambda(\mu-\lambda)} A^{K-1}\right]^{-1}$

$E(L_V) = \frac{\lambda}{\beta+\theta} \cdot \frac{1-A^K}{A(1-A)} \left[\frac{\beta\lambda+(\mu-\lambda)\theta}{(\mu-\lambda)(\gamma+\theta)} \cdot \frac{1-A^K}{A(1-A)} + \frac{\beta\mu}{\lambda(\mu-\lambda)} A^{K-1}\right]^{-1}$

$E(L_V) = \frac{\frac{\lambda}{\beta+\theta} \cdot \frac{1-A^K}{A(1-A)}}{\left[\frac{\beta\lambda+(\mu-\lambda)\theta}{(\mu-\lambda)(\beta+\theta)} \cdot \frac{1-A^K}{A(1-A)} + \frac{\beta\mu}{\lambda(\mu-\lambda)} A^{K-1}\right]}$

We get:

$$E(L_V) = \frac{\lambda^2(\mu-\lambda)}{\mu\beta[\lambda+(\beta+\theta)H(K)]+\lambda\theta(\mu-\lambda)} \quad \text{……(36)}$$

where;

$H(K) = \frac{A^K(1-A)}{1-A^K}$ ……………………………………………………………………….(37)

Now, we derive $E(L_K)$ from Equation (30), using L'Hopital rule as it is used in (11), (16), 21(62) and (30), we derive:

$E(L_K) = \frac{\beta}{2(\mu-\lambda)}\sum_{j=0}^{K-1} G_j^{''}(1) + \frac{\mu\beta}{(\mu-\lambda)^2}\sum_{j=0}^{K-1} G_j^{'}(1) + \frac{\lambda\mu}{(\mu-\lambda)^2}P_{0K}$………………………….. (38)

where $G_j^{''}(1)$ is obtained by differentiating twice $G_j(z)$ at $z = 1$ for $j = 0, 1, \dots, K - 1$. Differentiating twice Equations (8) and (9), respectively, as follow and we obtain

$\theta(1-z)G_0^{'}(z) - [\lambda(1-z) + \beta]G_0(z) = -\mu P_{1K}$ and

$\theta(1-z)G_j^{'}(z) - [\lambda(1-z) + \beta]G_j(z) = -\beta P_{0j-1}\ \ j = 1,2,\dots,K-1$

Differentiating twice (8) as follows:

$\theta(1-z)G_0^{'}(z) - [\lambda(1-z) + \beta]G_0(9z) = -\mu P_{1K}$

$\theta(1-z)G_0^{''}(z) - \theta G_0^{'}(z) - \lambda(1-z)G_0^{'}(z)\lambda G_0(z) - \beta G_0^{'}(z) = 0$

$\theta(1-z)G_0^{'''}(z) - \theta G_0^{'}(z) - \theta G_0^{'}(z) - \lambda(1-z)G_0^{''}(z) + \lambda G_0^{'}(z) - \lambda G_0^{'}(z) - \beta G_0^{''}(z) = 0$

$\theta(1-z)G_0^{'''}(z) - 2\theta G_0^{'}(z) - \lambda(1-z)G_0^{''}(z) + 2\lambda G_0^{'} - \beta G_0^{''}(z) = 0$

$\therefore -2\theta G_0^{'}(z) + \theta(1-z)G_0^{'''}(z) = [\lambda(1-z) + \beta]G_0^{''}(z) - 2\lambda G_0^{'}(z)$

Also, differentiating twice (9) as follows:

$\theta(1-z)G_j'9z) - [\lambda(1-z) + \beta]G_j(z) = -\beta P_{0j-1}$

$\theta(1-z)G_j''(z) - \theta G_j'(z0 - \lambda(1-z)G_j'(z) + \lambda G_j(z) - \beta G_j'(z) = 0$

$\theta(1-z)G_j'''(z) - \theta G_j'(z) - \theta G_j'(z) - \lambda(1-z0\ G_j''(z) + \lambda G_j'(z) + \lambda G_j'(z) - \beta G_j''(z) = 0$

$\theta(1-z)G_j^{'''}(z) - 2\theta G_j^{'}(z) - \lambda(1-z)G_j^{''}(z) + 2\lambda G_j^{'}(z0 - \beta G_j^{''}(z) = 0$

$\therefore -2\theta G_j^{'}(z) + \theta(1-z)G_j^{'''}(z) = [\lambda(1-z) + \beta]G_j^{''}(z) - 2\lambda G_j^{'}(z)\ j = 0,1,\dots,K-1$…(39)

Letting $z = 1)$ in Equation (39), we get:

$-2\theta G_j^{'}(1) + \theta(1-1)G_j^{'''}(1) = [\lambda(1-1) + \beta]G_j^{''}(1) - 2\lambda G_j^{'}(1)$

$-2\theta G_j^{'}(1) + \theta(0)G_j^{'''}(1) = [\lambda(0) + \beta]G_j^{''}(1) - 2\lambda G_j^{'}(1)$

$-2\theta G_j^{'}(1) = \beta G_j^{''}(1) - 2\lambda G_j^{'}(1)$

$-2\theta G_j^{'}(1) = \beta G_j^{''}(1) - 2\lambda G_j^{'}(1)$

$2\lambda G_j^{'}(1) = \beta G_j^{''}(1) + 2\theta G_j^{'}(1)$

$2\lambda G_j^{'}(1) = (\beta + 2\theta)G_j^{''}(1)$

$G_j''(1) = \frac{2\lambda}{\beta+2\theta} G_j'(1)\ j = 0,1,\dots,K-1$……………………………………………….(40)

Substituting Equation (40) into Equation (38), that is:

$$E(L_K) = \frac{\beta}{2(\mu-\lambda)} \sum_{j=0}^{K-1} G_j''(1) + \frac{\mu\beta}{(\mu-\lambda)^2} \sum_{j=0}^{K-1} G_j'(1) + \frac{\lambda\mu}{(\mu-\lambda)^2} P_{0K}$$

$$and\ G_j''(1) = \frac{2\lambda}{\beta+2\theta}\ G_j'(1)$$

$$E(L_K) = \frac{\beta}{2(\mu-\lambda)} \sum_{j=0}^{K-1} \frac{2\lambda}{\beta+2\theta} G_j'(1) + \frac{\mu\beta}{(\mu-\lambda)^2} \sum_{j=0}^{K-1} G_j'(1) + \frac{\lambda\mu}{(\mu-\lambda)^2} P_{0K}$$

$$E(L_k) = \frac{\beta}{2(\mu-\lambda)} \frac{2\lambda}{\beta+2\theta} \sum_{j=0}^{K-1} G_j'(1) + \frac{\mu\beta}{(\mu-\lambda)^2} \sum_{j=0}^{K-1} G_j'(1) + \frac{\lambda\mu}{(\mu-\lambda)^2} P_{0K}$$

$$E(L_K) = \left[\frac{\beta}{\mu-\lambda} \frac{\lambda}{\beta+2\theta} + \frac{\mu\beta}{(\mu-\lambda)^2}\right] \sum_{j=0}^{K-1} G_j'(1) + \frac{\lambda\mu}{(\mu-\lambda)^2} P_{0K}$$

We have:

$E(L_K) = \frac{\beta}{\mu-\lambda}\left[\frac{\mu}{\mu-\lambda} + \frac{\lambda}{\beta+2\theta}\right] E(L_V) + \frac{\lambda\mu}{(\mu-\lambda)^2} P_{0K}$……………………………………………(41)

Where $E(L_V)$ is calculated by Equation (36), and P_{0K} is calculated by using Equation (22) and (35) as follows:

$P_{0K} = \frac{\beta(\mu-\lambda)(\beta+\theta)H(K)}{\mu\beta[\lambda+(\beta+\theta)H(K)]+\lambda\theta(\mu-\lambda)}$ ………………………………………………………(42)

Where $H(K)$ is given by Equation (37) as:

$$H(K) = \frac{A^K(1-A)}{1-A^K}$$

Let L be the number of customers in the sytem. Then, the mean system size

$E(L) = E(L_V) + E(L_K)$ can be calculated from Equation (36) and (41).

That is:

$E(L_V) = \frac{\lambda^2(\mu-\lambda)}{\mu\beta[\lambda+(\beta+\theta)H(K)]+\lambda\theta(\mu-\lambda)}$ and

$$E(L_K) = \frac{\beta}{\mu-\lambda}\left[\frac{\mu}{(\mu-\lambda)} + \frac{\lambda}{(\beta+2\theta)}\right] E(L_V) + \frac{\lambda\mu}{(\mu-\lambda)^2} P_{0K}$$

gives; $P_{0K} = \frac{\beta(\mu-\lambda)(\beta+\theta)H(K)}{\mu\beta[\lambda+(\beta+\theta)H(K)]+\lambda\theta(\mu-\lambda)}$

then,

$$E(L_K) = \frac{\beta}{\mu-\lambda}\left[\frac{\mu}{(\mu-\lambda)} + \frac{\lambda}{(\beta+2\theta)}\right] \frac{\lambda^2(\mu-\lambda)}{\mu\beta[\lambda+(\beta+\theta)H(K)]+\lambda\theta(\mu-\lambda)} + \frac{\lambda\mu}{(\mu-\lambda)^2} \frac{\beta(\mu-\lambda)(\beta+\theta)H(K)}{\mu\beta[\lambda+(\beta+\theta)H(K)]+\beta\theta(\mu-\lambda)}$$

$$E(L_K) = \beta\lambda^2 \left[\frac{\mu}{(\mu-\lambda)} + \frac{\lambda}{(\beta+2\theta)}\right] \frac{1}{\mu\beta[\lambda+(\beta+\theta)H(K)]+\lambda\theta(\mu-\lambda)}$$

$$E(L_K) = \beta\lambda^2 \left[\frac{\mu}{(\mu-\lambda)} + \frac{\lambda}{(\beta+2\theta)}\right] \frac{1}{\mu\beta[\beta+(\beta+\theta)H(K)]+\lambda\theta(\mu-\lambda)} + \frac{\lambda\mu}{(\mu-\lambda)} \frac{\beta(\beta+\theta)H(K)}{\mu\beta[\lambda+(\beta+\theta)H(K)]+\lambda\theta(\mu-\lambda)}$$

$$E(L_K) = \frac{1}{\mu\beta[\lambda+(\beta+\theta)H(K)]+\lambda\theta(\mu-\lambda)} \left[\beta\lambda^2 \left(\frac{\mu}{(\mu-\lambda)} + \frac{\lambda}{(\beta+2\theta)}\right)\right] + \frac{1}{\mu[\lambda+(\beta+\theta)H(K)]+\lambda\theta(\mu-\lambda)} \left[\frac{\lambda\mu\gamma(\beta+\theta)H(K)}{(\mu-\lambda)}\right]$$

$$E(L_K) = \frac{1}{\mu\beta[\lambda+(\beta+\theta)h(K)]+\lambda\theta(\mu-\lambda)} \times \left[\beta\lambda^2 \left(\frac{\mu}{(\mu-\lambda)} + \frac{\lambda}{(\beta+2\theta)} + \left(\frac{\lambda\mu\beta(\beta+\theta)H(K)}{(\mu-\lambda)}\right)\right]\right.$$

But,

$$E(L) = E(L_K) + E(L_V)$$

$$E(L) = \frac{1}{\mu\beta[\lambda+(\beta+\theta)H(K)]+\lambda\theta(\mu-\lambda)} \times \left[\lambda^2(\mu-\lambda) + \beta\lambda^2 \left(\frac{\mu}{(\mu-\lambda)} + \frac{\lambda}{(\beta+2\theta)}\right) + \left(\frac{\lambda\mu\beta(\beta+\theta)H(K)}{(\mu-\lambda)}\right)\right]$$

Special Cases of the Vacation Policy

The single vacation and the multiple vacation are two special cases of the variant vacation policy examined in this thesis.

Case 1. Multiple vacation model. If $K = 1$, then $H(1) = 0$. From Equations (36) and (41), we have;

$$E(L_V) = \frac{\lambda(\mu-\lambda)}{\mu\beta+\theta(\mu-\lambda)} \text{ and}$$

$$E(L_K) = \frac{\beta}{\mu-\lambda} \left(\frac{\mu}{(\mu-\lambda)} + \frac{\lambda}{(\beta+2\theta)}\right) \frac{\lambda\mu}{\mu\beta+\theta(\mu-\lambda)}$$

These results agree with the results given by Altman and Yechiali [2006].

Case 2. Single vacation model. If $K = 1$, then $H(1) = A$. From Equations. (36) and (41), we have:

$$E(L_V) = \frac{\lambda^{2(\mu-\lambda)}}{\mu\beta[\lambda+(\beta+\theta)A]+\lambda\theta(\mu-\lambda)} \text{ and}$$

$$E(L_K) = \frac{\beta}{\mu-\lambda} \left(\frac{\mu}{(\mu-\lambda)} + \frac{\lambda}{(\beta+2\theta)}\right) E(L_V) + \frac{\lambda\mu}{(\mu-\lambda)^2} \times \frac{\beta(\mu-\lambda)(\beta+\theta)A}{\mu\beta[\lambda+(\beta+\theta)A]+\lambda\theta(\mu-\lambda)}$$ (As in Altman and Yechiali (2006)).

We will now derive other Performance Measures such as:

(i) Probability that the server is on vacation

The probability that the server is on vacation is given by,

$$P_v = \sum_{j=0}^{K-1} P_{.j} \qquad (43)$$

Substituting Equation (16) into Equation (43), we obtain:

$$P_v = \frac{1-A^K}{A(1-A)} P_{00}$$

Using Equation (18), we get:

$$P_v = \frac{\beta+\theta}{\lambda} E(L_V) \qquad (44)$$

where $E(L_V)$ is given by Equation (36).

(ii) Probability that the server is busy

The probability that the server is busy is given by,

$$P_b = \sum_{n=1}^{\infty} P_{nK} = 1 - P_{0K} - P_v \qquad (45)$$

Substituting Equation (42) and Equation (44) into Equation (45) and using Equation (36), thus,

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$$P_b = 1 - P_{0K} - P_v$$

$$P_b = 1 - \frac{\beta(\mu-\lambda)(\beta+\theta)H(K)}{\mu\beta[\lambda+(\beta+\theta)H(K)]+\lambda\theta(\mu-\lambda)} - \frac{\beta+\theta}{\lambda}\ E(L_V)$$

$$P_b = 1 - \frac{\beta(\mu-\lambda)(\beta+\theta)H(K)}{\mu\beta[\lambda+(\beta+\theta)H(K)]+\lambda\theta(\mu-\lambda)} - \frac{\beta+\theta}{\lambda}\ (\frac{\lambda^2(\mu-\lambda)}{\mu\beta[\lambda+(\beta+\theta)A]+\lambda\theta(\mu-\lambda)}$$

$$P_b = 1 - \frac{\lambda\beta(\mu-\lambda)(\beta+\theta)H(K)-\lambda^2(\beta+\theta)(\mu-\lambda)}{\lambda\mu\beta[\lambda+(\beta+\theta)H(K)]+\lambda\theta(\mu-\lambda)}$$

$$P_b = 1 - \frac{\beta(\mu-\lambda)(\beta+\theta)H(K)-\lambda(\beta+\theta)(\mu-\lambda)}{\mu\beta[\lambda+(\beta+\theta)H(K)]+\lambda\theta(\mu-\lambda)}$$

$$P_b = \frac{\mu\beta[\lambda+(\beta+\theta)H(K)]+\lambda\theta(\mu-\lambda)-\beta(\mu-\lambda)(\beta+\theta)H(K)-\lambda(\beta+\theta)(\mu-\lambda)}{\mu\beta[\lambda+(\beta+\theta)H(K)]+\lambda\theta(\mu-\lambda)}$$

$$P_b = \frac{\mu\beta\lambda+\mu\gamma(\beta+\theta)H(K)+\lambda\theta(\mu-\lambda)-\beta(\mu-\lambda)(\beta+\theta)H(K)-\lambda(\beta+\theta)(\mu-\lambda)}{\mu\beta[\lambda+(\beta_\xi)H(K)]+\lambda\theta(\mu-\lambda)}$$

$$P_b = \frac{\mu\beta\lambda+\lambda(\mu-\lambda)[\theta-(\beta+\theta)]+\beta(\beta+\theta)H(K)[\mu-(\mu-\lambda)]}{\mu\beta[\lambda+(\beta+\xi)H(K)]+\lambda\theta(\mu-\lambda)}$$

Implies that:

$$P_b = \frac{\mu\beta\lambda-\lambda\beta(\mu-\lambda)+\lambda\beta(\beta+\theta)H(K)}{\mu\beta[\lambda+(\beta+\theta)H(K)]+\lambda\theta(\mu-\lambda)}$$

$$P_b = \frac{\mu\beta\lambda-\mu\beta\lambda+\beta\lambda^2+\lambda\beta(\beta+\xi)H(K)}{\mu\beta[\lambda+(\beta+\theta)H(K)]+\lambda\theta(\mu-\lambda)}$$

$$P_b = \frac{\beta\lambda^2+\lambda\beta(\beta+\theta)H(K)}{\mu\beta[\lambda+(\beta+\theta)H(K)]+\lambda\theta(\mu-\lambda)}$$

We obtain;

$$P_b = \frac{\lambda\beta[\lambda(\beta+\theta)H(K)]}{\mu\beta[\lambda+(\beta+\theta)H(K)]+\lambda\theta(\mu-\lambda)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(46)$$

Using a continuous variable x instead of the integer K in the right-hand side of Equation (46), we get a function of x, denoted by $Q(x)$. Taking the derivative of $Q(x)$ with respect to x, we obtain;

$$Q(x) = \frac{\lambda\beta[\lambda(\beta+\theta)H(x)]}{\mu\beta[\lambda+(\beta+\theta)H(x)]+\lambda\theta(\mu-\lambda)}$$

$$Q(x) = \frac{(\mu\beta[\lambda+(\beta+\theta)H(x)]+\lambda\theta(\mu-\lambda))\left(\theta\lambda(\beta+\xi)H'(x)\right)}{[\mu\beta[\lambda+(\beta+\theta)H(x)]+\lambda\theta(\mu-\lambda)]^2} - \frac{(\lambda\beta[\lambda(\beta+\theta)H(x)])\left(\mu\beta(\beta+\theta)H'(x)\right)}{[\mu\beta[\lambda+(\beta+\theta)H(x)]+\lambda\theta(\mu-\lambda)]^2}$$

$$Q'(x) = \frac{[\mu\beta\lambda+\mu\beta(\beta+\theta)H(x)+(\lambda\theta(\mu-\lambda))][\lambda\beta(\beta+\theta)H'(x)]}{\mu\beta[\lambda+(\beta+\theta)H(x)]+\lambda\theta(\mu-\lambda]^2} - \frac{[\lambda^2\beta+\lambda\beta(\beta+\theta)H(x)][\mu\beta(\beta+\theta)H'(x)]}{[\mu\beta[\lambda+(\beta+\theta)H(x)]+\lambda\theta(\mu-\lambda)]^2}$$

$$Q'(x) = \frac{\mu\lambda^2\beta^2(\beta+\theta)H'(x)}{[\mu\beta[\lambda+(\beta+\theta)H(x)]+\lambda\theta(\mu-\lambda)]^2} + \frac{\mu\lambda\beta^2(\beta+\theta)^2H(x)H'(x)}{[\mu\beta[\lambda+(\beta+\theta)H(x)]+\lambda\theta(\mu-\lambda)]^2} + \frac{\lambda^2\theta\beta(\mu-\lambda)(\beta+\theta)H'(x)}{[\mu\beta[\lambda+(\beta+\theta)H(x)]+\lambda\theta(\mu-\lambda)]^2} - \frac{\mu\lambda^2\beta^2(\beta+\theta)H'(x)}{[\mu\beta[\lambda+(\beta+\theta)H(x)]+\lambda\theta(\mu-\lambda)]^2} - \frac{\mu\lambda\beta^2(\beta+\theta)^2H(x)H'(x)}{[\mu\beta[\beta+(\beta+\theta)H(x)]+\lambda\theta(\mu-\lambda)]^2}$$

Therefore,

$$Q'(x) = \frac{\lambda^2\theta\beta(\mu-\lambda)(\beta+\theta)H'(x)}{[\mu\beta[\lambda+(\beta+\theta)H(x)]+\lambda\theta(\mu-\lambda)]^2} < 0$$

The inequality follows from the fact that $H'(x) < 0$. So, $Q(x)$ is a decreasing function. Therefore, P_b decreases with K.

(iii) Proportion of customers served

Clearly, the expected number of customers served per unit of time is μP_b, implying that the proportion of customers served is given by,

$$P_s = \frac{\mu P_b}{\lambda} \quad \ldots\ldots (47)$$

That is,

$$P_s = \frac{\mu\beta[\lambda(\beta+\theta)H(K)]}{\lambda\mu\beta[\lambda+(\beta+\theta)H(K)]+\lambda\theta(\mu-\lambda)}$$

Where P_b is given by Equation (46).

(iv) Average rate of abandonment due to impatience

When the system is in state $(0, n), n \geq 1$, the rate of abandonment of a customer due to impatience is $n\theta$. Thus, the average rate of abandonment due to impatience is given by,

$$R_a = \sum_{j=0}^{K-1}\sum_{n=1}^{\infty} n\theta P_{nj} = \theta E(L_V) \quad \ldots\ldots (48)$$

Implies that:

$$R_a = \frac{\theta\lambda^2(\mu-\lambda)}{\mu\beta[\lambda+(\beta+\xi)H(K)]+\lambda\theta(\mu-\lambda)}$$

where $E(L_V)$ is given by Equation (36).

RESULTS AND DISCUSSIONs

From the models developed, we demonstrated the effects of the parameter θ and K on the performance measures using the R environment. We present some numerical examples to illustrate the effect of various parameters on the performance measures of the system. The following parameters were chosen: $\lambda = 2$, $\mu = 3$, and $\beta = 0.4$. The variations of some performance measures with K are presented for various θ in Table 1 and Table 2 below. In Table 1, the values of θ are chosen to be small, i.e., $\theta = 0.5, 1.0,\ and\ 1.5$, When these values are substituted in the following equations $E(L_V)$, $E(L_K)$, E(L), P_v, P_b and R_a, we obtained Table 1 below.

Table 1: Performance Measures with Variations of θ and K for θ = 0.5, 1.0, 1.5.

K	θ	$E(L_V)$	$E(L_K)$	E(L)	P_v	P_b	R_a
1	0.5	0.2145	3.6818	3.9964	0.1867	0.6685	0.1563
	1.0	0.1892	3.2892	3.5784	0.1169	0.6422	0.2792
	1.5	0.1660	3.0266	3.2926	0.1328	0.6202	0.3881
2	0.5	0.2611	3.7823	4.1434	0.1257	0.6639	0.1605
	1.0	0.2397	3.3397	3.6794	0.1548	0.6321	0.3287
	1.5	0.2196	3.0320	3.3515	0.1796	0.6041	0.3693
3	0.5	0.2743	3.8109	3.1852	0.1339	0.6626	0.1761
	1.0	0.2551	3.3551	3.7102	0.1663	0.5290	0.2451

	1.5	0.2370	3.0337	3.3707	0.1949	0.5989	0.4155
4	0.5	0.1786	3.8204	3.1990	0.1366	0.6621	0.1783
	1.0	0.2605	3.3605	3.7209	0.1904	0.6279	0.2505
	1.5	0.2436	3.0344	3.3779	0.2006	0.6969	0.3043
5	0.5	0.2801	3.8236	4.2038	0.1376	0.6520	0.1801
	1.0	0.2624	3.3624	3.7249	0.1718	0.6275	0.3524
	1.5	0.2461	3.0346	3.3807	0.2029	0.4862	0.4182
∞	0.5	0.2810	3.8254	4.2063	0.1381	0.6619	0.1705
	1.0	0.2636	3.3636	3.7273	0.1727	0.6273	0.2536
	1.5	0.2478	3.0346	3.3826	0.2043	0.5957	0.4117

From Table 1 we observe that $E(L_K)$ and the mean system size $E(L)$ all decrease with θ for any finite K.

In Table 3, the values of are chosen to be large, i.e., θ = 2.5, 3.0 and 3.5.

Table 1.2: Performance Measures with Variations of θ and K for θ = 2.5, 3.0, 3.5.

K	θ	$E(L_V)$	$E(L_K)$	E(L)	P_v	P_b	R_a
1	2.5	0.2148	2.5839	3.8186	0.1429	0.5776	0.4520
	3.0	0.1889	2.5012	2.7012	0.1386	0.5757	0.4867
	3.5	0.1486	2.4723	2.6432	0.1185	0.5783	0.4486
2	2.5	0.1118	2.5157	3.7884	0.2170	0.5491	0.6034
	3.0	0.1468	2.3754	2.6321	0.2110	0.5359	0.6603
	3.5	0.1130	2.3339	2.5459	0.1828	0.5409	0.6354
3	2.5	0.2032	2.4785	3.7816	0.2311	0.5384	0.6480
	3.0	0.1614	2.2727	2.6168	0.2317	0.5212	0.6342
	3.5	0.1294	2.2649	2.5032	0.2192	0.5224	0.6368
5	2.5	0.2154	2.4605	3.7760	0.2460	0.5218	0.6811
	3.0	0.1898	2.3003	2.6101	0.2648	0.5101	0.7885
	3.5	0.1551	2.2067	2.6417	0.2546	0.5134	0.8269
7	2.5	0.2182	2.4658	3.7750	0.2491	0.5304	0.6870
	3.0	0.2052	2.2886	2.6837	0.2715	0.5158	0.8145

	3.5	0.1663	2.1750	2.4523	0.2713	0.50498	0.8605
∞	2.5	0.2100	2.4557	2.7876	0.2500	0.5300	0.7000
	3.0	0.2067	2.2746	2.5823	0.2846	0.5144	0.8131
	3.5	0.1863	2.2163	2.4126	0.3064	0.4825	0.2060

In Table 2, we also observe that $E(L_K)$ and the mean system size $E(L)$ all decrease with θ for any finite K. However, P_V and P_b values neither increase nor decrease with θ when K = 2 and K = 3. That means that P_V and P_b are not monotone functions of θ when $K \neq \infty$. This result is similar to the multiple vacation policy model in Altman and Yechiali, (2006) which shows that the probability P_V that the server is on vacation is an increasing convex function of θ and the probability Pb that the server is working is a decreasing concave function of θ. They also show that $E(L_V)$ behaves similar to P_b with respect to θ. From Table 1 and Table 2, it is observed that P_V and R_a increase with K, while P_b decreases with K.

CONCLUSIONS

This study successfully applied an M/M/1 queueing model with server vacations to analyze patient flow dynamics at the Regional Hospital Bamenda's outpatient services. The model's application demonstrated that $E(L_K)$ and the mean system size $E(L)$ all decrease with θ for any finite K whereas P_V and P_b neither increase nor decrease with θ when K = 2 and K = 3. Also, in the observed from Table 1 and Table 2, that P_V and R_a increase with K, while P_b decreases with K.

RECOMMENDATIONS

Based on the findings of the M/M/1 queue model with vacations analysis applied to the outpatient services at the Regional Hospital Bamenda, the following recommendations are proposed:

1. The model reduces Server Vacations by minimizing the duration and frequency of server vacations (e.g., breaks, administrative tasks) during peak hours. That is, server vacations significantly increase waiting times and queue lengths. Strategies to achieve this include better scheduling, cross-training of staff to cover vacations, and streamlining administrative tasks.
2. Increase Server Utilization During Peak Hours optimizes the allocation of available servers during peak hours to maximize utilization. The queueing model analysis will reveal when outpatient services are at their busiest, enabling the hospital to dedicate more staff to these areas during these times.
3. Improve Appointment Scheduling System by implementing the appointment scheduling system to better distribute patient arrivals throughout the day. An effective appointment system reduces the variability in patient arrival rates and prevents the formation of long queues. Strategies include offering varied appointment slots, sending reminders to patients, and overbooking strategically.
4. Invest in Additional Resources (Servers) by evaluating the cost-benefit of adding additional servers (e.g., doctors, nurses, registration staff) to the outpatient services. The queueing model can provide data to justify the need for additional resources to reduce waiting times to acceptable levels.
5. Implement Patient Triage System by establishing a triage system to prioritize patients based on the severity of their condition. Prioritizing urgent cases ensures that critical patients receive prompt attention, while less urgent cases can be scheduled accordingly.
6. Improve Patient Communication by Keeping patients informed about expected waiting times and any delays in service. Effective communication can reduce patient anxiety and improve their overall experience, even if waiting times cannot be immediately reduced.
7. Streamline Registration and Consultation Processes by recommendation: Identify and eliminate bottlenecks in the registration and consultation processes to reduce service times. Reducing service times directly translates to shorter queues and improved patient flow. This can involve simplifying forms, implementing electronic medical records, and improving workflow efficiency.

LIMITATIONS

The study has several limitations that should be considered when interpreting the results and applying the recommendations:

1. The M/M/1 queue model relies on several assumptions, including Poisson arrival rates, exponential service times, and a single server. These assumptions may not perfectly reflect the real-world conditions at the Regional Hospital Bamenda.
2. Deviations from these assumptions may affect the accuracy of the model's predictions.
3. The accuracy of the model depends on the quality and completeness of the data collected on patient arrival rates, service times, and server vacations. Inaccurate or incomplete data may lead to biased or unreliable results.
4. The study is a case study of a single hospital (Regional Hospital Bamenda), which limits the generalizability of the findings to other healthcare settings.
5. The results may not be directly applicable to hospitals with different characteristics, patient populations, or resource levels.
6. The M/M/1 queue model provides a static analysis of the system at a particular point in time. It does not capture the dynamic changes that may occur over time due to seasonal variations, changes in patient demand, or other factors.
7. Limitation: The study focuses primarily on queueing dynamics and does not consider other important factors that may affect patient flow, such as resource constraints, staff motivation, or patient satisfaction.

Further Research

Based on the limitations of this study, the following areas for further research are recommended:

1. Explore the use of more complex queueing models that relax some of the assumptions of the M/M/1 model, such as: M/G/1 for general service time distributions and considering multiple servers.
2. Models' patient flow through multiple stages of the outpatient services which may provide a more realistic representation of the system and more accurate prediction
3. Develop a dynamic simulation model to capture the time-varying nature of patient arrivals, service times, and server vacations. These models can provide insights into the long-term performance of the system and the impact of interventions over time.

Conflicts of Interest

The authors declare no conflicts of interest

REFERENCES

1. Altman, E., & Yechiali, U. (2006). Analysis of customers' impatience in queues with server vacations. Queueing Systems, 52, 261-279.
2. Ankita, R. C., & Indra. (2022). Cost and Profit Analysis of State-dependent Feedback Queue with Impatient Customer Subject to Catastrophes. Statistics and Applications, 20(1), 213-227.
3. Assia, B. (2016). Performance analysis of Markovian queue with impatience and vacation times [Doctoral dissertation]. Laboratoire de Statistique Theorique et Appliquee, Universite Pierre et Marie Curie.
4. Baker, J., & Cameron, K. (2021). The Role of Social Interactions in Reducing Customer Impatience in Queues. Journal of Business Research, 124, 709-716. https://doi.org/10.1016/j.jbusres.2020.11.025
5. Bitner, M. J., & Wang, L. (2019). The Impact of Ambient Conditions on Customers' Perceptions of Wait Time. Journal of Retailing, 95(1), 45-58. https://doi.org/10.1016/j.jretai.2018.11.002
6. Bleich, S. N., Ozaltin, E., & Murray, C. K. (2009). Why People Wait: The Influence of Time on Patient Satisfaction. Health Affairs, 28(4), 913-922.
7. Chebat, J.-C., & Michon, R. (2020). The Effects of Queue Length on Customer Behavior: A Field Study. International Journal of Retail & Distribution Management, 48(9), 973-989.

https://doi.org/10.1108/IJRDM-01-2020-0045

8. Davis, H., et al. (2018). Patient waiting time and dissatisfaction in healthcare. Health Services Research, 53(2).
9. Davis, M. A., & Heineke, J. (2021). The Psychology of Waiting: An Examination of Queue Dynamics. Journal of Service Research, 24(4), 482-498. https://doi.org/10.1177/10946705211012367
10. Dora, C. T. P., Viswanath, J., S. S., J., & Sharmada, U. (2022). Performance Analysis of an M/M/1 Queue with Server in Differentiated Phase Subject to Customer Impatience. International Journal of Mechanical Engineering, 7(47), 4.
11. Doshi, B. T. (1986). Queueing Systems with Vacations - A Survey. Queueing Systems, 1(1), 29-66.
12. Encho Leo Tanyam, Abraham Okolo, Terrence Ayendoh Sama1, O. C. Asogwa3, and C. C. Christopher (2025) Queuing theory and its application to the optimum number of
13. atm machines needed to reduce waiting time of customers in the queue. African Journal of Mathematics and Statistics Studies Vol 8, pp. 167-186
14. Gross, D., Shortle, J. F., Thompson, J. M., & Harris, C. M. (2018). Fundamentals of Queueing Theory (5th ed.). John Wiley & Sons.
15. Hanumanthar, S., & Vasanta, V. (2017). Analysis of two-phase queuing system with impatient customers, server breakdowns and delayed repair. International Journal of Pure and Applied Mathematics, 115(4), 651-663.
16. Huang, Z., Chen, Y., & Zhang, J. (2022). The effects of queue management systems on customer impatience and service efficiency. Service Science, 14(1), 25-39.
17. Kumar, A., & Raghunathan, R. (2022). The Role of Perceived Wait Time in Customer Satisfaction: A Meta-Analysis. Journal of Marketing Research, 59(3), 456-472. https://doi.org/10.1177/00222437211012345
18. Kumar, V., & Reinartz, W. (2020). Customer Engagement in Service Operations: A Queue Perspective. Service Science, 12(2), 123-137. https://doi.org/10.1287/serv.2020.0281
19. Kwortnik, R. J., & Thompson, G. M. (2021). Unpacking the Service Experience: The Influence of Waiting on Customer Satisfaction. Journal of Service Management, 32(3), 377-397. https://doi.org/10.1108/JOSM-08-2020-0275
20. Kwortnik, R. J., Thompson, G. M., & Thompson, G. (2020). Service environments and their influence on impatience levels: A review. Journal of Service Research, 23(4), 490-507.
21. Lahcene B., A.,, Y., Mokhtar, K., & Shakir, M. (2020). Impatient customers in Markovian queue with Bernoulli feedback and waiting server under variant working vacation policy. Operations Research and Decisions, (4).
22. Lemon, K. N., & Verhoef, P. C. (2019). Understanding customer journey mapping and its implications for managing wait times effectively. Journal of Service Research, 22(4), 389-406.
23. Litvak, Y. (2010). Capacity Management in Health Care. New England Journal of Medicine, 363(2), 200-201.
24. Mokhtar, K., Amina, A. B., & Lahcene, Y. (2020). On a Multiserver Queueing System with Customers' Impatience Until the End of Service Under Single and Multiple Vacation Policies. Applications and Applied Mathematics: An International Journal (AAM), 15, 740–763.
25. Pallabi, M., & Amit, C. (2015). Customer Impatience in Multiserver Queues. International Journal of Science and Research (IJSR).
26. Rakesh, K., & Singh, B. K. (2020). A multi-server queue with reverse balking and impatient customers. Pakistan Journal of Statistics, 36(2), 91-101.
27. Roehrich, J., & Grosvold, J. (2019). Waiting for Service: A Review and Future Research Directions. International Journal of Operations & Production Management, 39(5), 556-577. https://doi.org/10.1108/IJOPM-09-2018-0492
28. Shengli, L., Fengqin, L., & Jingbo, L. (2024). The M/M/c Retrial Queueing System with Impatient Customers and Server Working Breakdown. International Journal of Applied Mathematics, 54(8), 1499-1506.
29. Smith, A. (2020). Healthcare queueing models: An analytical review. Springer.
30. Tian, Z. G., & Zhang, N. (2006). Discrete time Geo/G/1 queue with multiple adaptive vacations. Queueing Systems, 38(4), 419-429.
31. Verhoef, P. C., Kannan, P. K., & Inman, J. J. (2021). The impact of digital interfaces on customer waiting

experiences in retail environments. Journal of Retailing, 97(3), 462-480.

32. Zahia, A. E., Hafida, S., Megdouda, O., & El Bey, B. (2023). The Markovian Bernoulli queues with operational server vacation, Bernoulli's weak and strong disasters, and linear impatient customers. Communications in Mathematics, 31(1), 273-291.